Multispecies Momentum Conserving Lorentz Collision Operator

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1. ABSTRACT

Why study impurities in magnetic confinement fusion?

- Detrimental to the plasma
- Predictive purposes, e.g. understanding impurity turbulent transport across the H-mode pedestal [1]
- Impurity density profiles impact ion temperature gradient driven turbulence transport [2, 3]

Why an accurate collision operator is crucial for studying impurities? Collision frequency scales as:

$$\nu_{\alpha\beta} \sim \frac{4\pi n_\beta q_\alpha^2 q_\beta^2 \ln\Lambda_{\alpha\beta}}{m_\alpha^2 v_{\text{th},\alpha}^3} \sim \frac{n_\beta q_\alpha^2 q_\beta^2}{\sqrt{m_\alpha}}.$$

e.g., for deuterium main ions and tungsten impurity:

$$\nu_{Z(i/e)} \approx 10\,\nu_{e(i/e)}, \qquad \nu_{i(i/e)} \approx 1/40\,\nu_{e(i/e)}$$

2A. LORENTZ OPERATOR

Derived from the Fokker-Planck/Rosenbluth operator:

$$C_{\alpha\beta}(f_\alpha) = \frac{\partial}{\partial \mathbf{v}} \cdot \left(-\frac{m_\alpha}{m_\alpha + m_\beta} \mathbf{K}_{\alpha\beta} f_\alpha + \frac{1}{2} \mathbf{D}_{\alpha\beta} \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} \right)$$

Assume heavy field particle with momentum restoring flow $\mathbf{u}_{\alpha\beta}$:

$$f_\beta = \delta^3(\mathbf{v} - \mathbf{u}_{\alpha\beta}), \quad m_\beta \gg m_\alpha$$

Using multipole expansion, the Lorentz collision operator reads:

$$C_{\alpha\beta}(f_\alpha) = \underbrace{\frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\nu(v)}{2} (v^2 \mathbb{1} - \mathbf{v}\mathbf{v}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} \right)}_{\text{Lorentz term: } C^T_{\alpha\beta}(f_\alpha)} + \underbrace{2\nu_{\alpha\beta} \mathscr{M}_\alpha \frac{\mathbf{u}_{\alpha\beta} \cdot \mathbf{v}}{v_{\text{th},\alpha}^2}}_{\text{Restoring term: } C^F_{\alpha\beta}(f_\alpha)}$$

Introduce the standard Lorentz collision operator $C^T_{\alpha\beta}(f_\alpha)$ conserves density and energy, **but not momentum**.

Idea: choose $\mathbf{u}_{\alpha\beta}$ such that momentum is conserved using:

$$u_{\alpha\beta} \left(\int 2\nu_{\alpha\beta} \mathscr{M}_\alpha \frac{v_\parallel}{v_{\text{th},\alpha}^2} \mathrm{d}^3\mathbf{v} \right) = \left(\int C^F_{\beta\alpha}(f_\beta) \mathrm{d}^3\mathbf{v} \right)$$

2B. RESTORING TERM

The Kovrizhnikh–Connor approach [4, 5] (same color terms cancel out):

$$m_\alpha n_\alpha \int C^T_{\alpha\beta}(f_\alpha) \mathrm{d}^3\mathbf{v} + m_\alpha n_\alpha \int C^F_{\alpha\beta}(f_\alpha) \mathrm{d}^3\mathbf{v} + m_\beta n_\beta \int C^T_{\beta\alpha}(f_\beta) \mathrm{d}^3\mathbf{v} + m_\beta n_\beta \int C^F_{\beta\alpha}(f_\beta) \mathrm{d}^3\mathbf{v} = 0$$

Problem: this operator restores momentum, but drives to the incorrect equilibrium.

Solution: choose $\mathbf{u}_{\alpha\beta}$ to recover expected friction force (e.g. from [6]):

$$m_\alpha n_\alpha \int C^T_{\alpha\beta}(f_\alpha) \mathrm{d}^3\mathbf{v} + \underbrace{m_\alpha n_\alpha \int C^F_{\alpha\beta}(f_\alpha) \mathrm{d}^3\mathbf{v}}_{R_{\alpha\beta} - m_\alpha n_\alpha \int C^T_{\alpha\beta}(f_\alpha) \mathrm{d}^3\mathbf{v}} + m_\beta n_\beta \int C^T_{\beta\alpha}(f_\beta) \mathrm{d}^3\mathbf{v} + \underbrace{m_\beta n_\beta \int C^F_{\beta\alpha}(f_\beta) \mathrm{d}^3\mathbf{v}}_{R_{\beta\alpha} - m_\beta n_\beta \int C^T_{\beta\alpha}(f_\beta) \mathrm{d}^3\mathbf{v}} = 0$$

3. IMPLEMENTATION

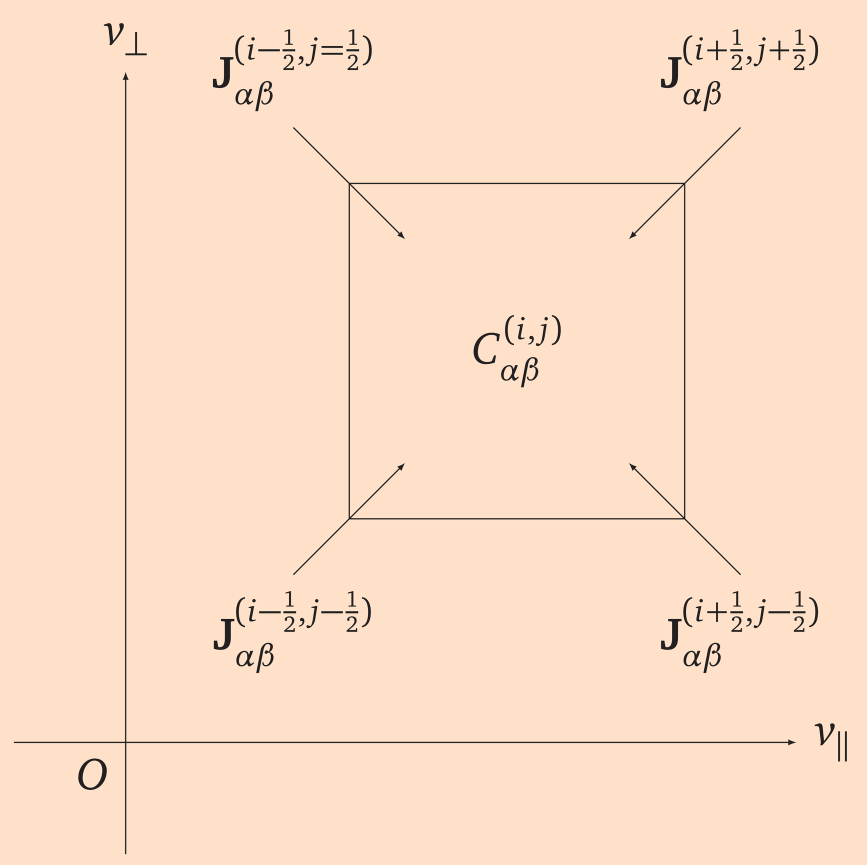

Figure 1: Finite volume scheme ensures conservation properties are fulfilled discretely

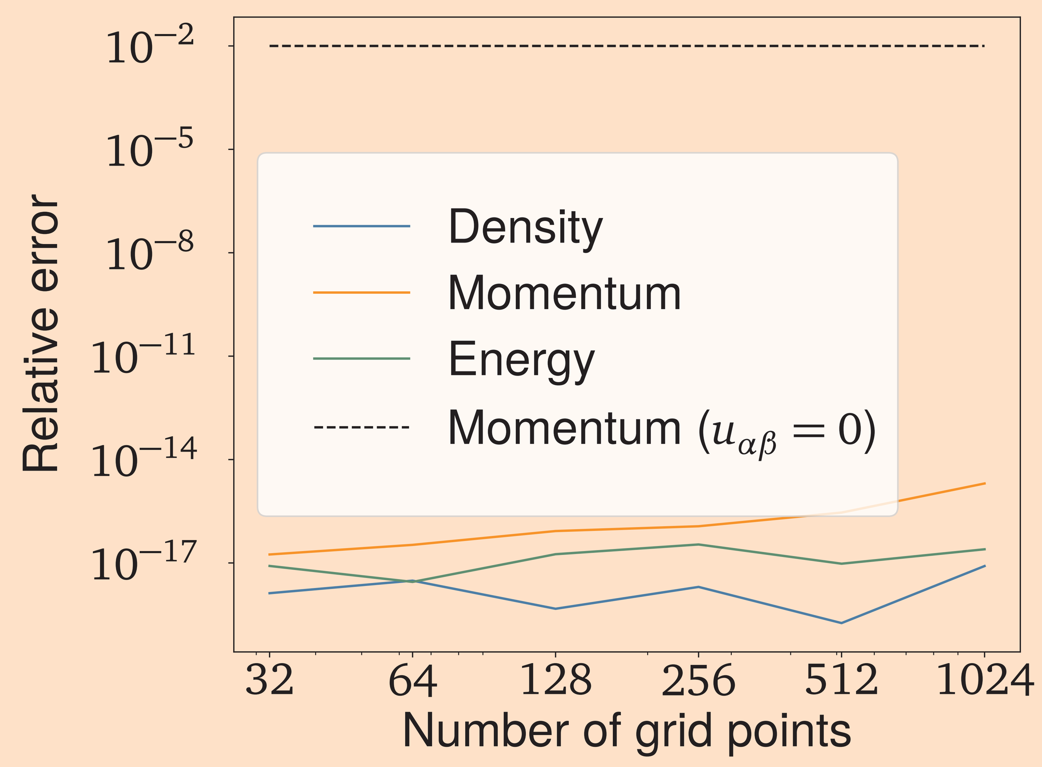

Figure 2: Verification of conservation properties with different number of grid points

4. RELAXATION STUDIES

Electron-deuterium initialized as bi-Maxwellians:

$$m_i = 2m_p,\ u_{e,0} = 60 \text{ km/s},\ u_{i,0} = 10 \text{ km/s}$$
$$T_{\parallel e,0} = 300 \text{ eV},\ T_{\perp e,0} = 390 \text{ eV},$$
$$T_{\parallel i,0} = 200 \text{ eV},\ T_{\perp i,0} = 260 \text{ eV}.$$

Analytical predictions:

$$\frac{\partial T_{\perp,\alpha}}{\partial t} = \frac{1}{2}\frac{\partial T_{\parallel,\alpha}}{\partial t} = \nu_{T,\alpha\beta}(T_{\perp,\alpha} - T_{\parallel,\alpha})$$
$$\frac{\partial u_\alpha}{\partial t} = \nu_{s,\alpha\beta}(u_\beta - u_\alpha)$$

Closed form solution:

$$T_\perp = T + \frac{1}{3}(T_{\perp,0} - T_{\parallel,0})\exp(-3\nu_T)$$
$$T_\parallel = T + \frac{2}{3}(T_{\parallel,0} - T_{\perp,0})\exp(-3\nu_T)$$
$$u_\alpha = u_E + (u_{\alpha,0} - u_E)\exp(-\nu_{s,\alpha\beta} t)\exp(-\nu_{s,\beta\alpha} t)$$

Flow relaxation properties of other models:

Connor: $\frac{\partial u_\alpha}{\partial t} = \nu_{s,\beta\alpha} u_\beta - \nu_{s,\alpha\beta} u_\alpha$

Without $C^F_{\alpha\beta}$: $\frac{\partial u_\alpha}{\partial t} = -\nu_s u_\alpha$

Comparison with literature [7]:

$$\nu^{\text{Lorentz}}_{s,e} = \frac{4}{3\sqrt{\pi}} \nu^{\text{th,Lorentz}}_e \approx \nu^{\text{NRL}}_{s,i}$$
$$\nu^{\text{Lorentz}}_{s,i} = \frac{4}{3\sqrt{\pi}} \frac{m_e}{m_i} \nu^{\text{th,Lorentz}}_e \approx \nu^{\text{NRL}}_{s,i}$$
$$\nu^{\text{Lorentz}}_{T,i} = \frac{8}{15\sqrt{\pi}} (\nu^{\text{th,Lorentz}}_{ee} + \nu^{\text{th,Lorentz}}_{ei}) \approx 2.8\,\nu^{\text{NRL}}_{T,e}$$
$$\nu^{\text{Lorentz}}_{T,i} = \frac{8}{15\sqrt{\pi}} (\nu^{\text{th,Lorentz}}_{ie} + \nu^{\text{th,Lorentz}}_{ii}) \approx 2.8\,\nu^{\text{NRL}}_{T,i}$$

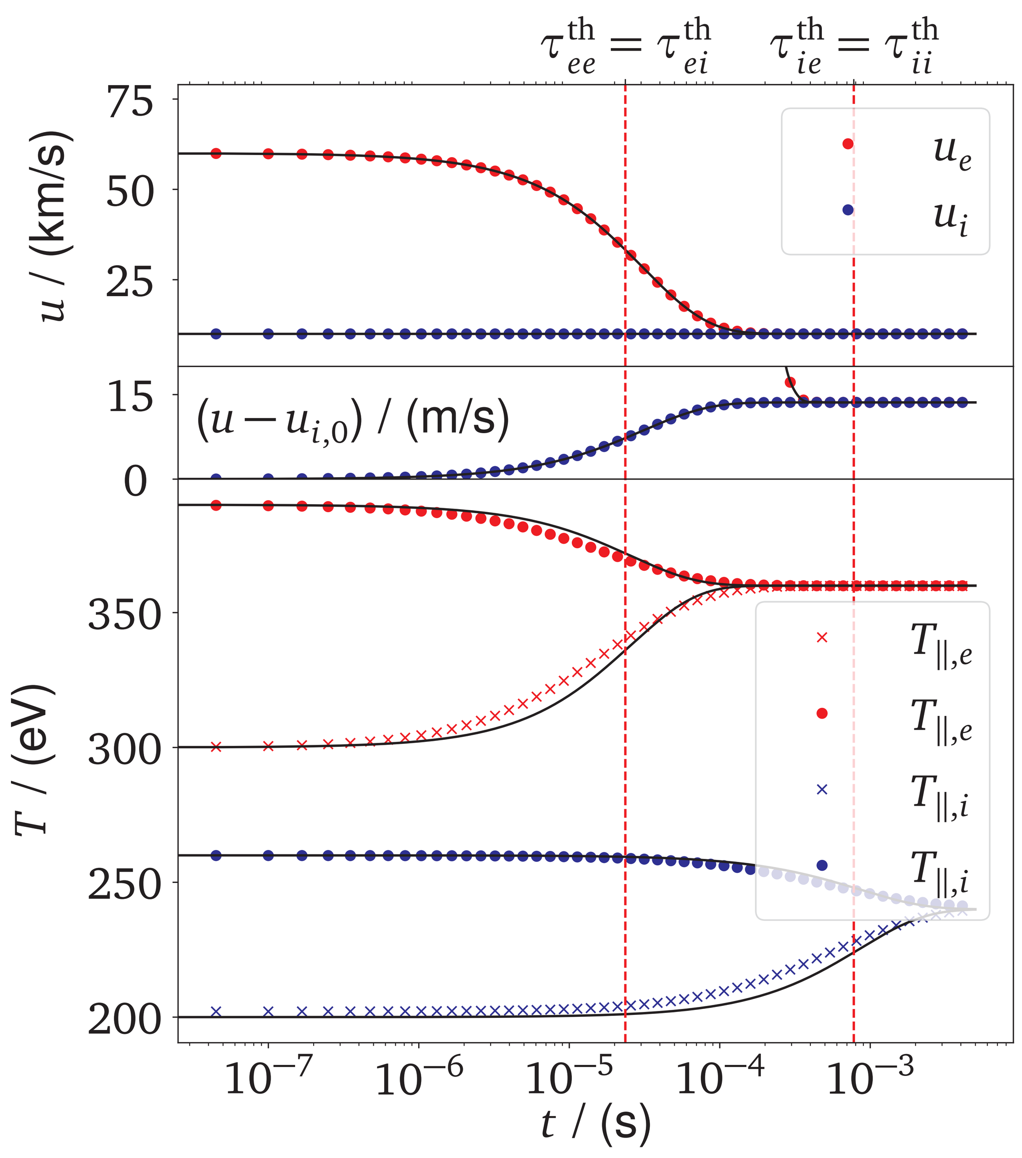

Figure 3: Comparison of analytical prediction (solid black lines) and numerical solution. Second figure zooms in the ion flow.

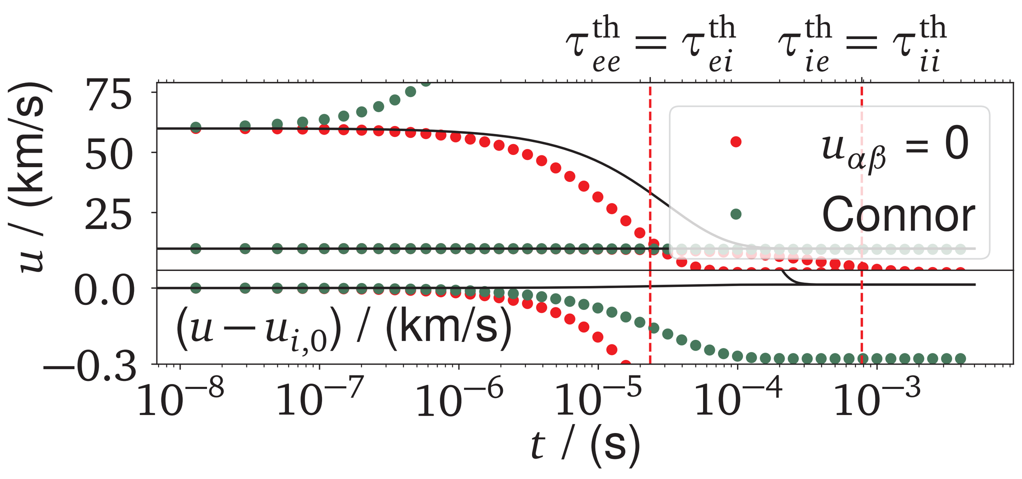

Figure 4: Relaxation properties of other models. Second figure zooms in the ion flow.

8. CONCLUSION

Key results:

- Development and implementation of a momentum restoring term for species with different flow velocity
- Verification of conservation and relaxation properties

Possible future works:

- Implementing more accurate collision operator, e.g. Hirshman-Sigmar [8] or Sugama [9] operator
- Simulating high collisional species, e.g. impurities

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