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### 4<sup>th</sup>-Order Accurate Methods for Relativistic MHD with Finite Conductivity

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• Astrophysical Context;

• Model Equations & Numerical Approach (Finite Volume, 2<sup>nd</sup> vs 4<sup>th</sup> order);

• Applications to Magnetic reconnection with effective resistivity;

• Conclusions.

# **Astrophysical Context**

### Jets from Active Galactic Nuclei (AGN)

- Powerful jets are produced in the central regions of (some) radio-loud Active Galactic Nuclei (AGN);
- Characterized by <u>non-thermal</u> emission from the radio to the X-rays and γ bands;
- Extends from a few Kpc to some Mpc;



• Convincing evidence of supersonic relativistic flows propagating in partially ordered magnetic fields.

## **Observations of AGN Jets**

#### • Direct observations of radio-galaxies:

- $_{\odot}\;$  radio luminosity that is  $10^{39}$   $10^{44}\; ergs/s;$
- Polarization degree 1% 30%;
- $_{\circ}~$  size from a few kpc to some Mpc;
- $_{\circ}~$  the morphological brightness distribution
- $_{\circ}\;$  polarization degree of the radio emission.

#### • By indirect means:

- $_{\odot}~$  life timescale,  $10^{7}$   $10^{8}~yrs$
- $_{\odot}~$  mean magnetic field, 10 10  $^{3}$   $\mu G$
- $_{\odot}\,$  kinetic power,  $10^{42}$   $10^{47}\,ergs/s.$









### **AGN Jets: Emission**

• Spectral energy distribution (SED) features two broad humps:



- Strong variability on timescales  $\leq$  day  $\rightarrow$  very compact emission regions where a sizeable fraction of the jet energy flux must be dissipated.
- Part of this energy becomes available to accelerate particles to ultra-relativistic energies.

### **Dissipation at the Small Scales**

Zooming at smaller scales, dissipation mechanisms may operate such as

- *<u>Collisionless relativistic shocks</u>:* dissipate kinetic energy into heat very efficiently.
  - prominent sites for particle acceleration through Fermi 1<sup>st</sup> order process.
  - Efficiency limited to almost || field geometries or weakly magnetized flows<sup>1</sup>;
- **<u>Relativistic magnetic reconnection</u>**: more promising candidate for producing high-energy particles and powering jet emission at small-scales<sup>2</sup>.
- <u>Velocity Shear</u>: particles can gain energy by scattering off small-scale magnetic field irregularities within the turbulent velocity layer at the jet / ambient interface<sup>3</sup>.



### **Model Equations & Numerical Approach**

#### Equation Model: Resistive Relativistic MHD

 For η≠0, the equations of (resistive) relativistic MHD (ResRMHD) derived from baryon number conservation, total momentum-energy conservation and Maxwell's equations (Ampere's law):

$$\begin{aligned} \nabla_{\mu}(\rho \mathbf{u}^{\mu}) &= 0 \,, \quad \nabla_{\mu} \left( T_{g}^{\mu\nu} + T_{EM}^{\mu\nu} \right) = 0 \\ \nabla_{\mu} F^{\mu\nu} &= -J^{\nu} \,, \quad \nabla_{\mu} F^{*\mu\nu} = 0 \end{aligned}$$

where  $D = \gamma \rho$  Mass density  $\mathcal{E} = w\gamma^2 - p + \mathcal{P}_{EM}$  Energy Density  $m = w\gamma u + E \times B$  Momentum Density  $T = -EE - BB + \frac{1}{2}(E^2 + B^2)$  Maxwell Stress  $J = \frac{1}{\eta}[\gamma E + u \times B - (E \cdot u)v] + qv$  Current Density [J'=  $\sigma$  E']  $\nabla \cdot B = 0, \quad \nabla \cdot E = q$  Constraints

$$\begin{aligned} \frac{\partial D}{\partial t} + \nabla \cdot (Dv) &= 0, \\ \frac{\partial m}{\partial t} + \nabla \cdot (wuu + pl + T) &= 0, \\ \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot m &= 0, \\ \frac{\partial B}{\partial t} + \nabla \times E &= 0, \\ \frac{\partial E}{\partial t} - \nabla \times B &= -J, \end{aligned}$$

### Equation Model: Resistive Relativistic MHD

- ResRMHD eqns admits 10 propagating modes\*, easily recognized in the small/large conductivity limits:
  - for η → ∞, matter and EM fields decouple; solution modes → pairs of light and acoustic waves (+ purely damped modes);
  - for η → 0 (ideal) limit, modes → pair of fast magnetosonic / slow / Alfvén modes. The contact mode unaffected by the conductivity.

! Important: resistivity is *collisional* in origin:



$$=rac{c^2}{4\pi\sigma}\,, \quad ext{where} \quad \sigma=rac{e^2n_e}{m_e
u_{ep}}$$

 $\eta$ 

#### Numerical method: Finite Volume Formulation

• We employ finite volume, so that integrating the previous differential equations over a control volume, yields (for zone-centered variables):

$$\frac{\partial U}{\partial t} = -\nabla \cdot F + S \implies \frac{d \langle U \rangle}{dt} = \underbrace{-\oint F \cdot dS}_{V} + \frac{1}{\Delta V} \int S dV$$
  
where 
$$-\left(\frac{\hat{F}_{x,\mathbf{x}_{f}} - \hat{F}_{x,\mathbf{x}_{f} - \hat{\mathbf{e}}_{x}}}{\Delta x} + \frac{\hat{F}_{y,\mathbf{y}_{f}} - \hat{F}_{y,\mathbf{y}_{f} - \hat{\mathbf{e}}_{y}}}{\Delta y} + \frac{\hat{F}_{z,\mathbf{z}_{f}} - \hat{F}_{z,\mathbf{z}_{f} - \hat{\mathbf{e}}_{z}}}{\Delta z}\right)$$

$$\langle U \rangle_c \equiv \frac{1}{\Delta x \Delta y \Delta z} \int U(x, y, z, t) \, dx \, dy \, dz$$

and  $U=(D, m, \mathcal{E}, \mathbf{E})$  is an array of conserved variables

• This is called the *integral form* of the equations.



$$\begin{split} \hat{F}_{x,\mathbf{x}_{f}} &\equiv \frac{1}{\Delta y \Delta z} \int \hat{\mathbf{e}}_{x} \cdot \mathsf{F} \left( U(x_{i+\frac{1}{2}}, y, z, t) \right) dy dz \,, \\ \hat{F}_{y,\mathbf{y}_{f}} &\equiv \frac{1}{\Delta x \Delta z} \int \hat{\mathbf{e}}_{y} \cdot \mathsf{F} \left( U(x, y_{j+\frac{1}{2}}, z, t) \right) dz dx \,, \\ \hat{F}_{z,\mathbf{z}_{f}} &\equiv \frac{1}{\Delta x \Delta y} \int \hat{\mathbf{e}}_{z} \cdot \mathsf{F} \left( U(x, y, z_{k+\frac{1}{2}}, t) \right) dx dy \,. \end{split}$$

### **Finite Volume + Constrained Transport**

• Magnetic fields retains a staggered representation (face-centered) and it is evolved through a discrete version of the Stokes theorem,

$$\begin{aligned} \frac{d\hat{B}_{x,\mathbf{x}_{f}}}{dt} &= -\left(\frac{\bar{E}_{z,\mathbf{z}_{e}} - \bar{E}_{z,\mathbf{z}_{e} - \hat{\mathbf{e}}_{y}}}{\Delta y} - \frac{\bar{E}_{y,\mathbf{y}_{e}} - \bar{E}_{y,\mathbf{y}_{e} - \hat{\mathbf{e}}_{z}}}{\Delta z}\right) \\ \frac{d\hat{B}_{y,\mathbf{y}_{f}}}{dt} &= -\left(\frac{\bar{E}_{x,\mathbf{x}_{e}} - \bar{E}_{x,\mathbf{x}_{e} - \hat{\mathbf{e}}_{z}}}{\Delta z} - \frac{\bar{E}_{z,\mathbf{z}_{e}} - \bar{E}_{z,\mathbf{z}_{e} - \hat{\mathbf{e}}_{x}}}{\Delta x}\right) \\ \frac{d\hat{B}_{z,\mathbf{z}_{f}}}{dt} &= -\left(\frac{\bar{E}_{y,\mathbf{y}_{e}} - \bar{E}_{y,\mathbf{y}_{e} - \hat{\mathbf{e}}_{x}}}{\Delta x} - \frac{\bar{E}_{x,\mathbf{x}_{e} - \hat{\mathbf{e}}_{y}}}{\Delta y}\right) \\ \\ \text{where, e.g.} \qquad \hat{B}_{x,x_{\mathrm{f}}} = \frac{1}{\Delta y \Delta z} \int \mathbf{B}(x_{i+\frac{1}{2}}, y_{j}, z_{k}) \cdot d\mathbf{S}_{x} \end{aligned}$$

• This ensure that *\\$\P\$*•*B*=0 condition is respected to machine accuracy.



$$\begin{split} \bar{E}_{x,\mathbf{x}_{e}} &\equiv \frac{1}{\Delta x} \int E_{x}(x,y_{j+\frac{1}{2}},z_{k+\frac{1}{2}},t) \, dx \,, \\ \bar{E}_{y,\mathbf{y}_{e}} &\equiv \frac{1}{\Delta y} \int E_{y}(x_{i+\frac{1}{2}},y,z_{k+\frac{1}{2}},t) \, dy \,, \\ \bar{E}_{z,\mathbf{z}_{e}} &\equiv \frac{1}{\Delta z} \int E_{z}(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},z,t) \, dz \,. \end{split}$$

### Flux Computation → Riemann Solver



• Fluxes computed by solving the so-called "*Riemann Problem*", i.e., the evolution of a discontinuity separating two *constant* states:



• Solution always considered to be discontinuous at cell interfaces: different level of approximation can be used\*.

 $\rightarrow$ 

• In 1D

### Extension to 4<sup>th</sup>-Order Method

- We draw on the original formulation by McCorquodale & Colella (2011)\* and \*\*.
- Volume average and point value interchangeable only at  $2^{nd}$  order:  $\langle U \rangle_c U_c = O(h^2)$

• To 4<sup>th</sup> order, e.g. 
$$U_c = \left(1 - \frac{\Delta}{24}\right) \langle U \rangle_c + O(h^4)$$

with 
$$\Delta \langle U \rangle_{\boldsymbol{c}} \equiv \left( \Delta^{x} + \Delta^{y} + \Delta^{z} \right) \langle U \rangle_{\boldsymbol{c}}$$

- "De-averaging" using Laplacian operators:  $\Delta^x \langle U \rangle_c = (\langle U \rangle_{c \hat{\mathbf{e}}_x} 2 \langle U \rangle_c + \langle U \rangle_{c + \hat{\mathbf{e}}_x})$
- Averaging follows the inverse rule, e.g.  $\hat{F}_{x,\mathbf{x}_f} = F_{x,\mathbf{x}_f} + \frac{\Delta_{\perp}^x F_{x,\mathbf{x}_f}}{24}$   $\bar{E}_{z,z_e} = \left(1 + \frac{\Delta^z}{24}\right) E_{z,z_e}$
- In its simplest form, a 4<sup>th</sup> order scheme can be designed by retaining the typical dimension by dimension strategy while relying on 1D operators.

### **Time Stepping: Handling Stiffness**

• Time evolution based on semi-discrete approach (method of lines), with stiff source ( $\eta \ll 1$ ) term

$$\frac{d\langle U\rangle_c}{dt} = -\oint \mathbf{F} \cdot d\mathbf{A} + \langle S\rangle_c \equiv R + \underbrace{S}_{c}$$

•  $\rightarrow$  <u>IM</u>plicit-<u>Ex</u>plicit (IMEX) RK methods\*:

 $\frac{\partial D}{\partial t} + \nabla \cdot (Dv) = 0,$  $\frac{\partial m}{\partial t} + \nabla \cdot (wuu + pl + T) = 0,$  $\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot m = 0,$  $\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0,$  $\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = \underbrace{-\mathbf{J} \equiv -\frac{\mathbf{\tilde{E}}}{\eta}}_{\eta}$ 

| IMEX-SSP3(4,3,3) L-sta  | able [*]   |
|---|--|
| $\langle U \rangle_{\boldsymbol{c}}^{(k)} = \langle U \rangle_{\boldsymbol{c}}^{\boldsymbol{n}} + \Delta t \sum_{i=1}^{k-1} \tilde{a}_{kj} \hat{R}_{\boldsymbol{c}} + \Delta t \sum_{i=1}^{k} a_{kj} \langle S \rangle_{\boldsymbol{c}}^{(j)},$ | Butcher Tableaux for SSP3(4,3,3):                      |
| $\langle U \rangle_{\boldsymbol{c}}^{n+1} = \langle U \rangle_{\boldsymbol{c}}^{n} + \Delta t \sum_{j=1}^{\nu} \tilde{w}_{j} \hat{R}_{\boldsymbol{c}}^{(j)} + \Delta t \sum_{j=1}^{\nu} w_{j} \langle S \rangle_{\boldsymbol{c}}^{(j)},$        | $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ |

#### Accuracy Assessment: CP Alfvé<u>n Waves</u>

• Circularly polarized Alfvén waves on  $[0,1]^3$ ,  $\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0 \\ -c_A \eta \cos \phi \\ -c_A \eta \sin \phi \end{pmatrix}$ ,  $\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} B_0 \\ B_0 \eta \cos \phi \\ B_0 \eta \sin \phi \end{pmatrix}$   $\phi = \mathbf{k}' \cdot \mathbf{x}' - \omega \mathbf{t}$ 



<u>CPU time saving</u>: for 2<sup>nd</sup> and 4<sup>th</sup> order to achieve the same accuracy, N<sub>2</sub> and N<sub>4</sub> are related by

$$N_4 \sim \left(\frac{C_4}{C_2}\right)^{1/4} \sqrt{N_2} \qquad \Rightarrow \quad \text{in terms of CPU time} \quad \Rightarrow \quad \frac{T_4}{T_2} = \left(\frac{C_4}{C_2}\right)^{(d+1)/4} \frac{\tau_4}{\tau_2} \frac{1}{\sqrt{N_2^{d+1}}} \frac{$$

$$C_4/C_2 \sim \{5, 8.7\}, \tau_4/\tau_2 \sim \{4.3, 3.7\} \rightarrow T_4/T_2 \sim K/N_2^2 \text{ with } K \sim \{21, 32\}$$

[\*] Berta et al., JCP., 499, 112701 (2024); [\*\*] Mignone et al, MNRAS (accepted)

#### Application to Relativistic Reconnection: Ideal Tearing Mode

• Ideal tearing mode in *ResRMHD*, with FF equilibrium

$$\mathbf{B} = B_0 \left[ \tanh\left(\frac{x}{a}\right) \hat{\mathbf{e}}_y + \operatorname{sech}\left(\frac{x}{a}\right) \hat{\mathbf{e}}_z \right] \qquad S = v_a L/\eta$$

- For sufficiently thin CS,  $a = S^{-1/3}L \rightarrow$  the reconnection process occurs on the ideal Alfvénic time scale:  $\tau_a = L/v_a$
- In the limit of large S  $\rightarrow$  *ideal* tearing instability [\*],[\*\*]
- Linear growth converges faster for  $4^{th}$  order scheme  $\rightarrow N_4 \leq N_2/2$



## **Collisionless Effective Resistivity**

### **Collisionless Resistivity: 2D PIC Models**

• Selvi et al.<sup>1</sup> (2023) analyzed 2D current sheets with PIC simulations of pair plasmas;



• At X-points, diverging flows result in a nondiagonal thermal pressure tensor: finite residence time for particles gives rise to a localized *collisionless effective resistivity*.

### **Effective Resistivity from PIC 2D Models**

 $\rightarrow$  Statistical analysis of Ohm's law to identify nonideal electric field contributions. Each species *s=e,p* (e.g. electron/positron) contributes:

$$oldsymbol{E} = -rac{\langleoldsymbol{v}_t
angle}{c} imes oldsymbol{B} + \sum_s rac{
ho_s}{n_t e_s} \left[ \partial_t ig\langleoldsymbol{u}_s
angle + \langleoldsymbol{v}_s
angle \cdot 
abla ig\langleoldsymbol{u}_s
angle 
ight] + \sum_s rac{1}{n_t e_s} 
abla \cdot \mathcal{P}_s$$

E = [ideal] + [temporal] + [convective] + [thermal]

| $n_t$ | = | $n_e + n_p$ | Total number density |  |
|-------|---|-------------|----------------------|--|
|-------|---|-------------|----------------------|--|

Four Velocity  $oldsymbol{u}_s~=~\gamma_soldsymbol{v}_s$ 

Thermal pressure tensor

ram pressure tensor

• Dominant contribution given by non-gyrotropic thermal pressure term.

### The Non-Ideal Electric Field at an X-point





### **Effective Resistivity in Fluid Model**

• Reformulated in terms of the spatial current density in the fluid frame,

$$\eta_{eff} = \frac{mc\partial_y v_y}{e\sqrt{\left(\frac{\rho ec}{m}\right)^2 - j_z^2}} \Rightarrow \eta_{eff} = \frac{\sqrt{\left(\frac{mc}{e}\partial_y v_y\right)^2 + e_z^2}}{\frac{\rho ec}{m}}$$

Dimensionless resistivity

$$ar{\eta}_{eff} = rac{1}{ar{
ho}} rac{\delta_0}{L_0} \sqrt{ar{e}_z^2 + \left(rac{\delta_0}{L_0}
ight)^2 \left(\partial_{ar{y}}ar{v}_y
ight)^2}$$

with 
$$\delta_0=c/\omega_0=c/\sqrt{rac{4\pi e^2
ho_0}{m^2}}$$
 and  $L_0,
ho_0$  scale quantities.

- e<sub>z</sub> = γ(E + v x B)<sub>z</sub> is the rest-frame electric field.
- Nonuniform nature of the effective resistivity may give a 1<sup>st</sup> approximation to collisionless reconnection in a (fluid) MHD description;
- No "free" parameters (as in Ripperda et al., 2019b);
- Dissipation is set by problem's scale and plasma properties.

### Validation: 1D Self-Similar Current Sheet

• Temporal evolution of a 1D current sheet: comparison between effective and constant resistivity:



### **2D Current Sheet with Effective Resistivity**

• 2D relativistic MHD models with non-uniform resistivity using both relativistic MHD (PLUTO code) and PIC (ZELTRON code).

• Initial condition:

• Parameters:

$$B_x(y) = B_0 anh\left(rac{y}{a}
ight) \,, \, p(y) = rac{1}{2}B_0^2(eta_0+1) - rac{1}{2}B_x^2(y) \,.$$

- Constant temperature  $\rightarrow \rho = \rho_0 p / p_0$
- $B_0 = \sqrt{\sigma_0 \rho_0}, \ \beta_0 = 0.01$
- $c_{A,0} = (1/\sigma_0 + 2\beta_0 + 1)^{-1/2}$
- $(x, y) \in [0, 4L] \times [-80\sqrt{\sigma_0}a, 80\sqrt{\sigma_0}a]$
- Periodic BC in x, reflective BC in y
- L = 1,  $a = 0.01L = \delta_0/2 = 5c/\omega_p$
- PIC benchmark performed with the ZELTRON code (Cerutti et al., 2013)

### **2D Current Sheet with Effective Resistivity**

• 2D relativistic MHD model with non-uniform resistivity.



### **Effective Resistivity Profile**



 $\rho l \rho_0$ 



#### **Effective vs. Constant Resistivity**



#### Reconnected flux and rate



 $\Phi_{\rm rec} = -\int_{x_X}^{x_O} B_y(x,0) dx = A_z \big|_X - A_z \big|_O$ 

### **Dependence on Grid Resolution**

• Effective  $\eta$  prescription captures the onset of magnetic reconnection even at low resolutions,  $N_x \approx 256$  (models with constant  $\eta$  require generally much higher resolutions to converge).

• Reconnecting magnetic fluxes in agreement for  $N_x \gtrsim 2048$  by the end of simulation, with the highest resolution case showing a slightly faster and more continuous reconnection.



### gPLUTO: the next GPU Version of the PLUTO Code

- 4<sup>th</sup> order method successfully implemented in gPLUTO the upcoming GPU version of the PLUTO code, developed within the SPACE CoE.
- SPACE (Scalable Parallel Astrophysical Codes for Exascale): Center of Excellence funded by the European HPC Joint Undertaking (JU).
- The CoE's primary objective is to prepare 7 of the existing state-ofthe-art European HPC astrophysics and cosmology codes for the transition to exa-scale on Euro HPC facilities.
- SPACE involves co-design activities bringing together scientists, code developers, HPC experts, HW manufacturers and SW developers.



#### Conclusions

• 4<sup>th</sup> order schemes deliver smaller dissipation, higher accuracy, CPU saving and more efficient computations;

• Non-uniform, effective resistivity model provides a viable opportunity to design physically grounded global models for reconnection-powered high-energy emission.

- Application to 2D reconnection: good agreement between PIC and ResRMHD models with effective resistivity:
  - Strong localization of magnetic dissipation within the current sheet
  - Dissipation set by the system's dynamics and introduction of a characteristic scale δ<sub>0</sub>.
  - Good results even at modest resolutions, while constant η case calls for much larger resolutions;



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# Thank You



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