Continuous Gravitational Waves: Methods, Results, and Practical Tools

Continuous gravitational waves and neutron stars workshop
AEI Hannover, Germany

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Overview

Continuous Wave Sources and Signal Model

Continuous Wave Search Methods

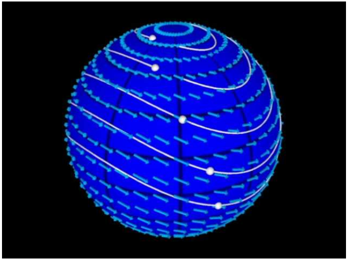
20 Years of Continuous Wave Search Results

Continuous Wave Software and Practical Examples

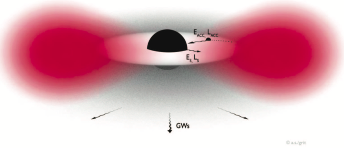
Continuous Wave Sources and Signal Model

Continuous Wave (CW) Sources

“Mountain” on a neutron star (with a spaceship flying over it and stars in the background, as suggested by Bing AI)



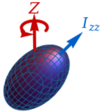
r-modes; current (instead of mass) quadrupole; see Fabian’s and Suprovo’s talks



Clouds of ultralight boson particles around black holes

Image credits: Microsoft Bing AI; University of Illinois at Urbana-Champaign; R. Brito et al., Classical and Quantum Gravity **32**, 134001 (2015)

“Mountains” on Neutron Stars (NSs)



Data analysis perspective: NS modelled as classical solid

- Fine for slowly rotating ($v \ll c$) stars
- Moment of inertia $\mathbf{I}$, principal moments I_{xx}, I_{yy}, I_{zz}

3 classes:

1. $I_{\mathrm{xx}} = I_{\mathrm{yy}} = I_{\mathrm{zz}}$: No gravitational waves
2. $I_{\mathrm{xx}} = I_{\mathrm{yy}} \neq I_{\mathrm{zz}}$: Biaxial rotator
   - *Ellipticity* $\epsilon = |I_{\mathrm{xx}} - I_{\mathrm{zz}}| / I_{\mathrm{zz}}$ assumed small
3. $I_{\mathrm{xx}} \neq I_{\mathrm{yy}} \neq I_{\mathrm{zz}}$: Triaxial rotator
   - Ellipticity $\epsilon = |I_{\mathrm{xx}} - I_{\mathrm{yy}}| / I_{\mathrm{zz}}$ assumed small
   - 2 subclasses:
     - Aligned: a principal axis (z) aligns with rotation axis (Z)
     - Nonaligned: no principal axis aligns with rotation axis

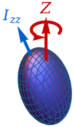
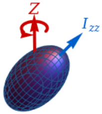


Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

“Mountains” on Neutron Stars (NSs)

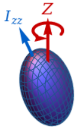
CW emission assumed quadrupolar; allowed frequencies:



- Rotation frequency[1] f_{rot}: NS has no rotational symmetry, rotation about non-principal axis
- $2f_{\mathrm{rot}}$: NS has 180° rotational symmetry, rotation about principal axis

For:

- $I_{\mathrm{xx}} = I_{\mathrm{yy}} \neq I_{\mathrm{zz}}$: Biaxial rotator
  - CW emission frequencies depends on angle θ between I_{zz} and rotation axis
  - CW emission at rotation frequency f_{rot} maximal at $\theta = 45°$, disappears at $\theta = 90°$
  - CW emission at $2f_{\mathrm{rot}}$ maximal at $\theta = 90°$



- $I_{\mathrm{xx}} \neq I_{\mathrm{yy}} \neq I_{\mathrm{zz}}$: Triaxial rotator
  - Aligned: CW emission at $2f_{\mathrm{rot}}$ only; *Most commonly assumed by CW searches*
  - Nonaligned: CW emission at f_{rot} and $2f_{\mathrm{rot}}$

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[1] possibly plus precession frequency

Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

CW Signal Model

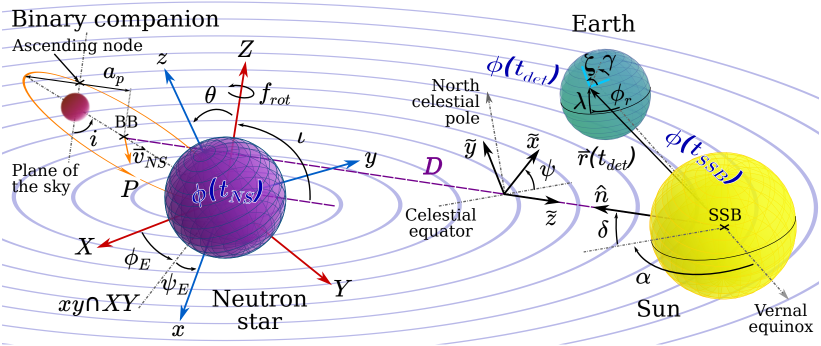
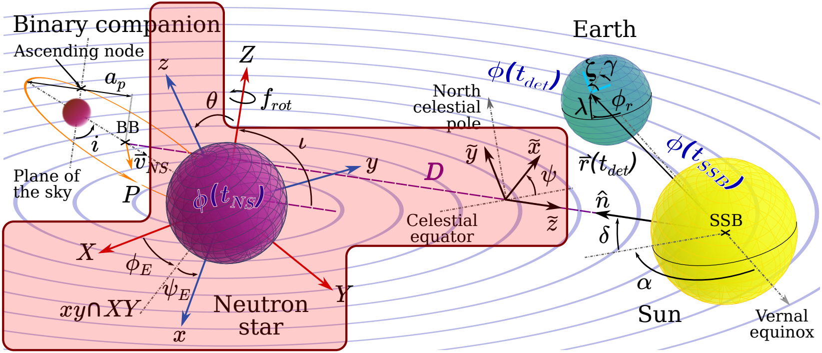


Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

CW Signal Model

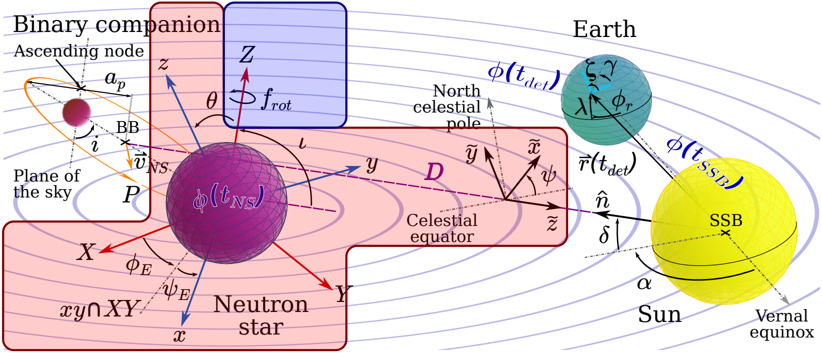


NS amplitude parameters

Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

CW Signal Model

NS frequency parameters



NS amplitude parameters

Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

CW Signal Model

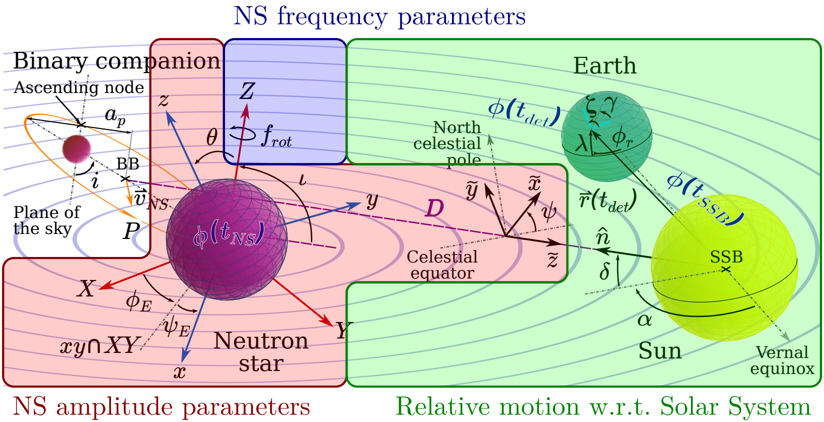
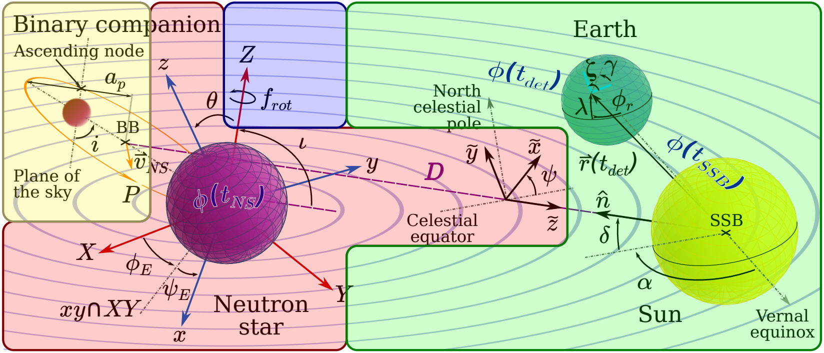


Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

CW Signal Model

Relative motion w.r.t. binary companion

NS frequency parameters



NS amplitude parameters

Relative motion w.r.t. Solar System

Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

CW Signal Model: Amplitude Parameters

- Strength of gravitational wave signal. Triaxial aligned model:

$$h_0 = \frac{4\pi^2 G}{c^4 D} I_{\mathrm{zz}} \epsilon f^2 \,, \tag{1}$$

where

- I_{zz}: principal moment
- $f = 2f_{\mathrm{rot}}$: CW frequency
- ϵ: ellipticity
- D: distance

- Orientation of NS relative to detector. Triaxial aligned model:
  - ι: Angle between NS rotation axis and line of sight to detector
    - Controls linear vs. circular polarisation of CW signal
  - ψ: Orientation of basis defining "+", "×" GW polarisation
  - ϕ_0: Arbitrary phase of CW signal at reference time
  - ι, ψ may be constrained by EM priors (pulsar wind nebulae)
  - No direct information about NS physics (without EM)
- Data analysis challenge: Easy
  - Find best-fit values from either analytic maximisation ($\mathcal{F}$-statistic), or Bayesian marginalisation (see Reinhard's talk)

CW Signal Model: Frequency Parameters

- Evolution of $f(t)$ unknown, but assumed to be slowly varying
- Can therefore model as Taylor series

$$f(t) = f_0 + f_1 (t - t_0) + \frac{1}{2} f_2 (t - t_0)^2 + \ldots \,, \tag{2}$$

  where
  - $f(t)$: Instantaneous CW frequency at time t
  - f_0: CW frequency at reference time t_0
  - $f_1 \equiv \dot{f}$: 1st *spindown* or frequency time derivative at t_0
  - $f_2 \equiv \ddot{f}$: 2nd *spindown* or frequency time derivative at t_0
    - 2nd spindowns typically only needed for "young" ($\lesssim$ 1 kyr) NSs
- Data analysis challenge: Hard
  - Need to perform "search": trial-and-error computation of detection statistic for discrete choices of f_0, f_1, etc.

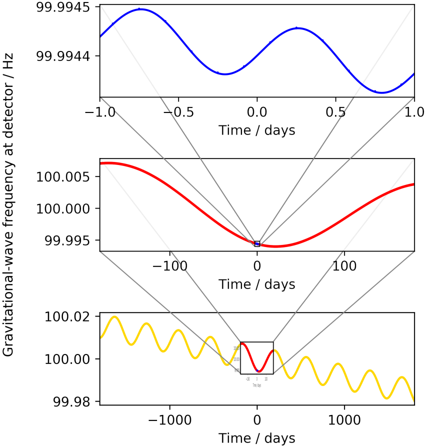
CW Signal Model: Relative Motion

Doppler modulation of CW signal; not a pure sinusoid

- Solar System: daily rotation of Earth, yearly orbit of Earth around Sun, parameters:
    - right ascension α
    - declination δ
- Binary companion (not shown), parameters:
    - orbital period P
    - projected semi-major axis a_p
    - if circular orbit: +1 parameter
    - if low-eccentricity orbit: +3 parameters

Data analysis challenge: Hard

- Need to search over parameters



Plus evolution of $f(t)$

Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

Continuous Wave Search Methods

GW Signals in Interferometric Detector

Detector measures time series[2]

$$
\begin{aligned}
h(t) &= F^{\mu\nu}(t) h_{\mu\nu}(t) && (3) \\
&= A_+(h_0, \cos\iota) F_+(t,\psi) \cos\left[\phi_0 + \phi(t,\vec{\lambda})\right] \\
&\quad + A_\times(h_0, \cos\iota) F_\times(t,\psi) \sin\left[\phi_0 + \phi(t,\vec{\lambda})\right] && (4)
\end{aligned}
$$

with $A_{+/\times}$ amplitudes, $F_{+/\times}$ antenna responses, ϕ CW signal phase

- Amplitude parameters (triaxial aligned model): h_0, $\cos\iota$, ψ, ϕ_0
- Phase parameters represented by $\vec{\lambda}$:
  - Evolution of $f(t)$: f_0, f_1, f_2, etc.
  - Doppler modulation from Earth motion: α, δ
  - Doppler modulation from binary companion: P, a_p, etc.

Reformat Eq. (4) as product of 4 amplitudes $\mathcal{A}^\mu$ and 4 functions h_μ [3]:

$$h(t) = \mathcal{A}^\mu(h_0, \cos\iota, \psi, \phi_0) h_\mu(t, \vec{\lambda}) \qquad (5)$$

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[2] Einstein summation over indices μ, ν in GW tensor $h_{\mu\nu}(t)$ and detector response $F^{\mu\nu}(t)$

[3] R. Prix, Physical Review D **75**, 023004 (2007).

CW Data Analysis: A Solved Problem?

In *principle*, yes: matched filtering[4]

- Construct log-likelihood ratio[5]

$$\log \mathcal{L} = \log \frac{P\big(\text{data } x(t) \big| x(t) = h(t) + \text{noise}\big)}{P\big(\text{data } x(t) \big| x(t) = \text{noise}\big)} \tag{6}$$

$$= \big(x(t) \big| h(t)\big) - \frac{1}{2}\big(h(t) \big| h(t)\big) \tag{7}$$

  assuming noise is Gaussian and (quasi-)stationary

- Functions $h_\mu(t, \vec{\lambda})$ are known for choice of $\vec{\lambda}$
- Amplitudes $\mathcal{A}^\mu$ can be analytically maximised over ($\mathcal{F}$-statistic)
- CW search algorithm:
  - Compute $\log \mathcal{L}$ (maximised over $\mathcal{A}^\mu$) for choices of $\vec{\lambda}$
  - Largest value of $\log \mathcal{L}$ found is our most promising candidate
  - Problem solved?!

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[4] P. Jaranowski et al., Physical Review D **58**, 063001 (1998).

[5] Notation: conditional probability $P(\text{statement} \,|\, \text{prior})$; inner product $\big(x(t) \,|\, y(t)\big)$ see Eq. (39) of Jaranowski et al.

CW Data Analysis: A Solved Problem?

In *practise*: no!!

- Except for CW searches targeting known pulsars, parameter space of possible $\vec{\lambda}$ is very^n large! ($n \gg 1$)
- Example: search of 1 year of data
  - all-sky survey (i.e. all α, δ)
  - frequency range $f_0 \in [50, 1000]$Hz
  - 1st spindown $f_1 \in [-10^{-8}, 0]$Hz/s
  - isolated NS (i.e. no binary orbit parameters)
- We would need to compute $\log \mathcal{L}$ for approximately
  - $\sim$ 9.9 trillion choices of (α, δ)
  - $\sim$ 40 billion choices of f_0
  - $\sim$ 2.3 million choices of f_1
  - So $\sim$ 920 octillion (10^{27}) values of $\log \mathcal{L}$ in total
- It takes[6] $\sim$ 810 microseconds to compute $\log \mathcal{L}$ for 1 year of data and *one* choice of parameters
- Total search time: $\sim$ 2.3 quintillion (10^{18}) years
  - e.g. $\sim$ 5 billion computers running non-stop for $\sim$ 5 billion years

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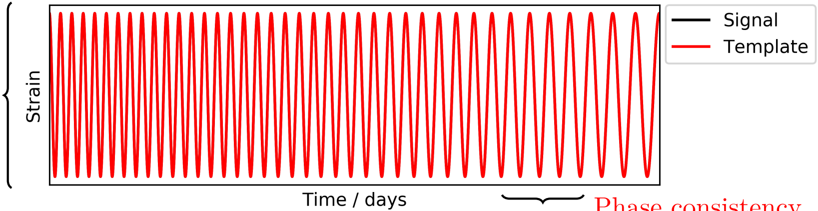
[6] This assumes the "demodulation" $\mathcal{F}$-statistic algorithm

The Semi-Coherent Paradigm

Fully coherent filter

Amplitude consistency

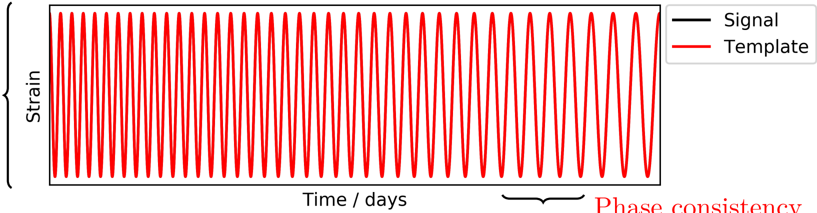


Phase consistency

The Semi-Coherent Paradigm

Fully coherent filter

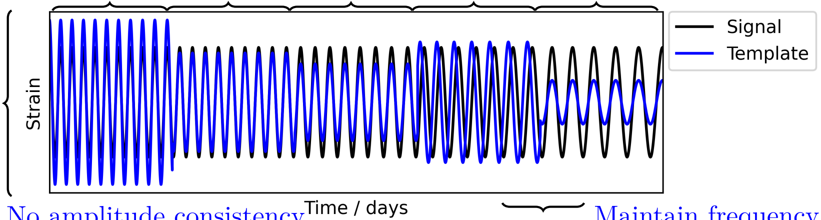
Amplitude consistency



Phase consistency

Semi-coherent filter

Segment Segment Segment Segment Segment



No amplitude consistency
☞ constant phase jumps

Maintain frequency
consistency

The Semi-Coherent Paradigm

- Fully coherent search:
  - Template is amplitude and phase consistent with signal
  - Computational cost $\propto$ (observation time)^m
    - $m \gg 1$
  - Sensitivity $\propto$ (observation time)$^{1/2}$
- Semi-coherent search:
  - Divide observation time into segments
  - Coherent stage:
    - Fully coherent search of each segment individually
    - Amplitude parameters may change between segments
  - Incoherent stage:
    - Combine fully-coherent filters from each segment together
    - Maintains frequency consistency over observation
  - Computational cost $\propto$ (# of segments)^n(segment time)^m
    - (segment time) $\ll$ (observation time)
    - $n \ll m$ [7]
  - Sensitivity $\propto$ (# of segments)$^{\sim 1/4}$(segment time)$^{1/2}$

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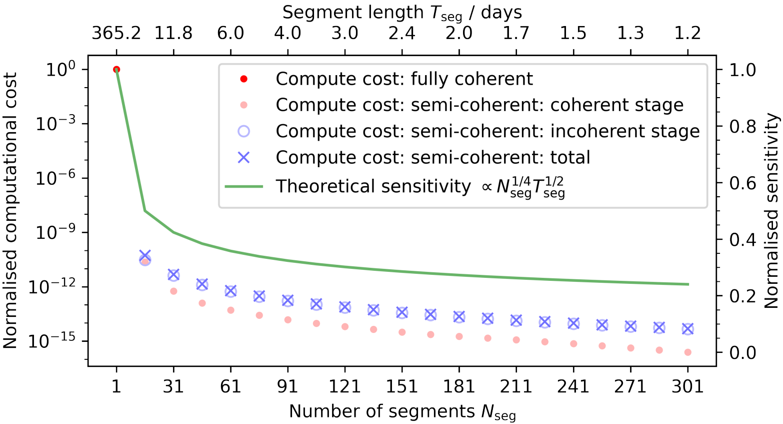
[7] R. Prix and M. Shaltev, Physical Review D **85**, 084010 (2012).

The Semi-Coherent Paradigm

Modelled computational cost of a semi-coherent CW search pipeline[8] analysing 1 year of data



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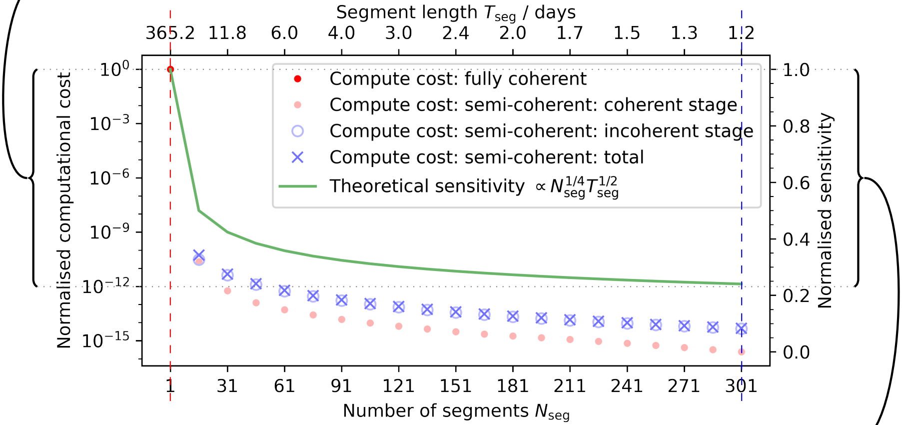
[8] Weave; K. Wette et al., Physical Review D **97**, 123016 (2018)

The Semi-Coherent Paradigm

Modelled computational cost of a semi-coherent CW search pipeline[8] analysing 1 year of data

Fully coherent 5× more sensitive, $10^{12}\times$ more expensive



Semi-coherent $10^{12}\times$ less expensive, 5× less sensitive

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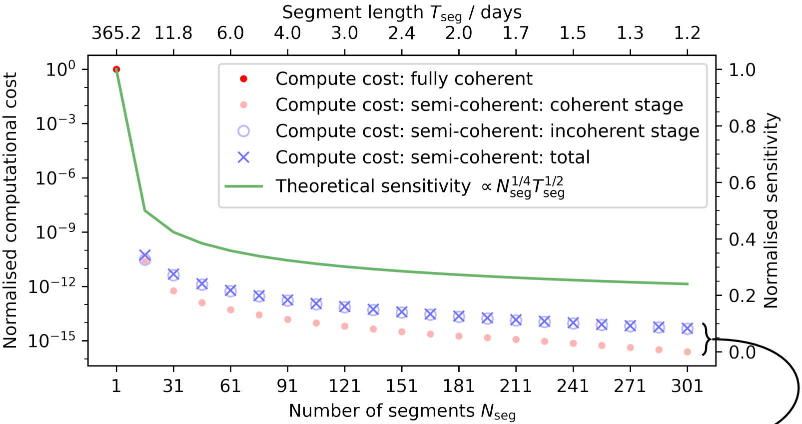
[8] Weave; K. Wette et al., Physical Review D **97**, 123016 (2018)

The Semi-Coherent Paradigm

Modelled computational cost of a semi-coherent CW search pipeline[8] analysing 1 year of data



Cost of incoherent stage dominates over coherent stage

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[8] Weave; K. Wette et al., Physical Review D **97**, 123016 (2018)

Coherent Algorithms

If $T_{\text{seg}} \ll 1\,\text{d}$:

- Frequency domain power:
  - Pre-compute Fourier transform of segment data
  - Power in a single bin
  - Data format/frameworks: Short Fourier Transforms (SFTs)[9], Short Fourier DataBase (SFDB), Band Sampled Data (BSD)[10]

If $T_{\text{seg}} \gtrsim 1\,\text{d}$:

- 5-vector method[11]:
  - Assume all phase parameters are accounted for
  - Earth rotation splits CW signal into 5 harmonics
  - Sum harmonics at $0, \pm 1, \pm 2$ Earth rotation frequency
- $\mathcal{F}$-statistic[12]:
  - Log-likelihood ratio maximised over $\mathcal{A}$ amplitudes

Fully coherent: heterodyning used for fully-coherent algorithms (e.g. Bayesian inference[13] over $\mathcal{A}$ amplitudes)

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[9] B. Allen et al., tech. rep. T040164-v2 (LIGO, 2022) [10] O. J. Piccinni et al., Classical and Quantum Gravity **36**, 015008 (2019) [11] P. Astone et al., Classical and Quantum Gravity **27**, 194016 (2010) [12] P. Jaranowski et al., Physical Review D **58**, 063001 (1998) [13] M. Pitkin, Journal of Open Source Software **7**, 4568 (2022)

Incoherent Algorithms

(Loosely) Categorised by detection statistic computed:

- None:
  - Coincidence: self-consistency between segments, inexpensive
- (Usually) Thresholded statistics:
  - Hough[14] family: 2D pattern matching (see Badri's talk)
- Summed statistics:
  - StackSlide[15]: unweighted, often used with $\mathcal{F}$-statistic
    Algorithms for combining self-consistent $\mathcal{F}$-statistics:
    - GCT[16]: based on studying large-scale correlations
    - Weave[17]: parameter space metric, optimal lattices
  - PowerFlux[18]: weighted by antenna response functions

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[14] B. Krishnan et al., Physical Review D **70**, 082001 (2004) [15] P. R. Brady and T. Creighton, Physical Review D **61**, 082001 (2000) [16] H. J. Pletsch and B. Allen, Physical Review Letters **103**, 181102 (2009) [17] K. Wette et al., Physical Review D **97**, 123016 (2018) [18] V. Dergachev, tech. rep. T1000272-v5 (LIGO, 2011)

Incoherent Algorithms

(Loosely) Categorised by signal model assumed:

- Tunable coherence: correlate segments at different times
  - Loosely coherent[19]: all-sky surveys
  - Cross-correlation[20]: directed targets esp. Scorpius X-1
- Stochastic frequency wandering:
  - Viterbi[21]: solves Hidden Markov Model
  - SOAP[22]: tunable memory
- Binary specialists:
  - TwoSpect[23]: looks for patterns in spectrograms
  - BinarySkyHough[24]: extension of Hough method
  - BinarySkyHou$\mathcal{F}$[25]: uses $\mathcal{F}$-statistic
  - BinaryWeave[26]: extension of Weave; see Arunava's talk

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[19] V. Dergachev, Classical and Quantum Gravity **27**, 205017 (2010) [20] J. T. Whelan et al., Physical Review D **91**, 102005 (2015) [21] S. Suvorova et al., Physical Review D **93**, 123009 (2016) [22] J. Bayley et al., Physical Review D **100**, 023006 (2019) [23] E. Goetz and K. Riles, Classical and Quantum Gravity **28**, 215006 (2011) [24] P. B. Covas and A. M. Sintes, Physical Review D **99**, 124019 (2019) [25] P. B. Covas and R. Prix, Physical Review D **106**, 084035 (2022) [26] A. Mukherjee et al., Physical Review D **107**, 062005 (2023)

$\mathcal{F}$-statistic

- Log-likelihood ratio[27]

$$\log\mathcal{L} = \bigl(x(t)\big|h(t)\bigr) - \frac{1}{2}\bigl(h(t)\big|h(t)\bigr) \tag{8}$$

$$= \vec{\mathcal{A}}\cdot\vec{x} - \frac{1}{2}\vec{\mathcal{A}}\cdot\boldsymbol{\mathcal{M}}\cdot\vec{\mathcal{A}} \tag{9}$$

  where $x_\mu = \bigl(x(t)\big|h_\mu(t)\bigr)$, $\mathcal{M}_{\mu\nu} = \bigl(h_\mu(t)\big|h_\nu(t)\bigr)$

- $\log\mathcal{L}$ is quadratic in $\vec{\mathcal{A}}$, analytic maximisation:

$$\frac{\partial\log\mathcal{L}}{\partial\mathcal{A}} = 0 \text{ is solved by } \vec{\mathcal{A}} = \vec{\mathcal{A}}_{\mathrm{MLE}} = \boldsymbol{\mathcal{M}}^{-1}\cdot\vec{x} \tag{10}$$

$$2\mathcal{F} = 2\log\mathcal{L}\big|_{\vec{\mathcal{A}}=\vec{\mathcal{A}}_{\mathrm{MLE}}} = \vec{x}\cdot\boldsymbol{\mathcal{M}}^{-1}\cdot\vec{x} \tag{11}$$

- *Note*: despite being named $\boldsymbol{\mathcal{F}}$-statistic[28], *value* of statistic is always denoted $\mathbf{2}\boldsymbol{\mathcal{F}}$

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[27] Notation: $(\cdot)$ represents contraction over neighbouring vector/matrix dimensions

[28] P. Jaranowski et al., Physical Review D **58**, 063001 (1998).

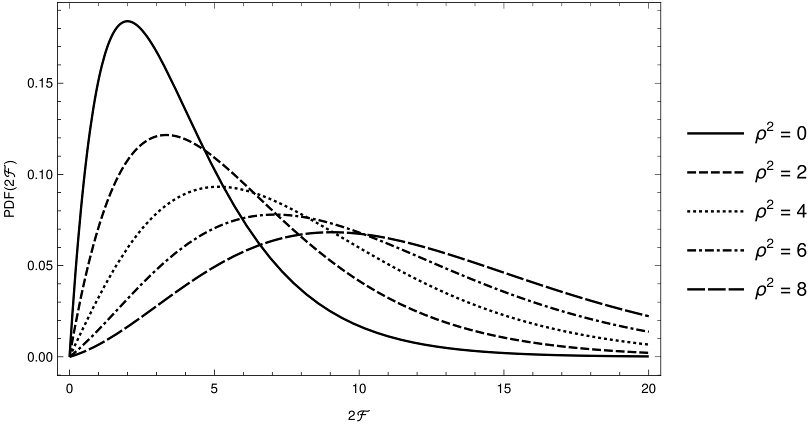
$\mathcal{F}$-statistic

Probability distribution:

- Can show $\vec{x}$ follows a multivariate normal distribution with means $\vec{\mu} = \boldsymbol{\mathcal{M}} \cdot \vec{\mathcal{A}}$ and covariance matrix $\boldsymbol{\mathcal{M}}$
  - $h_\mu(t)$ are not mutually orthogonal
- Cholesky decomposition $\boldsymbol{\mathcal{W}} \cdot \boldsymbol{\mathcal{W}}^T = \boldsymbol{\mathcal{M}}^{-1}$
- $\vec{y} = \boldsymbol{\mathcal{W}}^T \cdot \vec{x}$ are uncorrelated normal variables with means $\vec{\nu} = \boldsymbol{\mathcal{W}}^T \cdot \vec{\mu}$ and unit variances
- $2\mathcal{F} = \vec{x} \cdot \boldsymbol{\mathcal{M}}^{-1} \cdot \vec{x} = \|\vec{y}\|^2$ follows a χ^2 distribution with 4 degrees of freedom
  - y_μ are normal, y_μ^2 are χ^2 with 1 degree of freedom
  - Noncentrality parameter $\rho^2 = \|\vec{\nu}\|^2 = \vec{\mathcal{A}} \cdot \boldsymbol{\mathcal{M}} \cdot \vec{\mathcal{A}}$ is optimal signal-to-noise ratio of CW signal

$\mathcal{F}$-statistic

Probability distribution:



20 Years of Continuous Wave Search Results

Sensitivity vs Parameter Space/Computational Cost

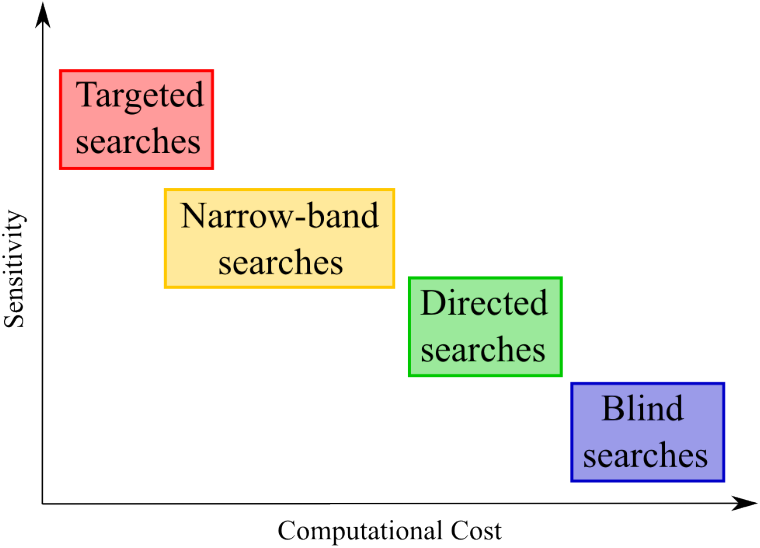
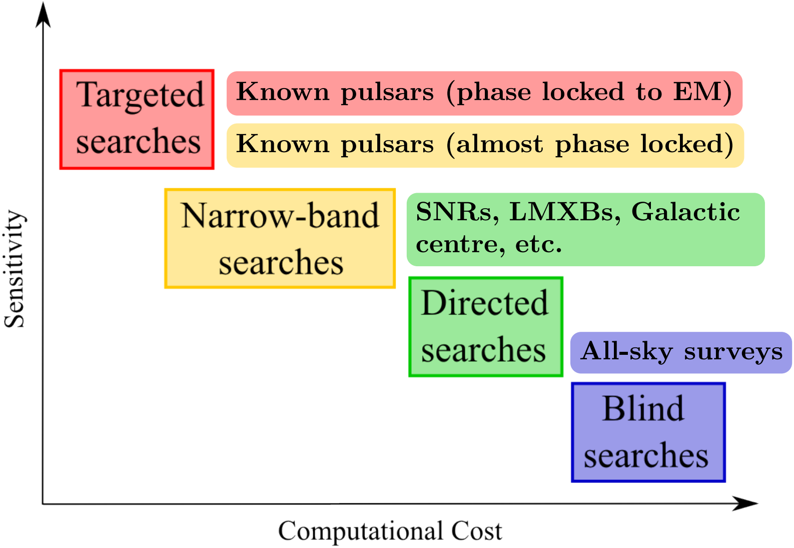


Image credits: M. Sieniawska and M. Bejger, Universe **5**, 217 (2019)

Sensitivity vs Parameter Space/Computational Cost



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Sensitivity vs Parameter Space/Computational Cost

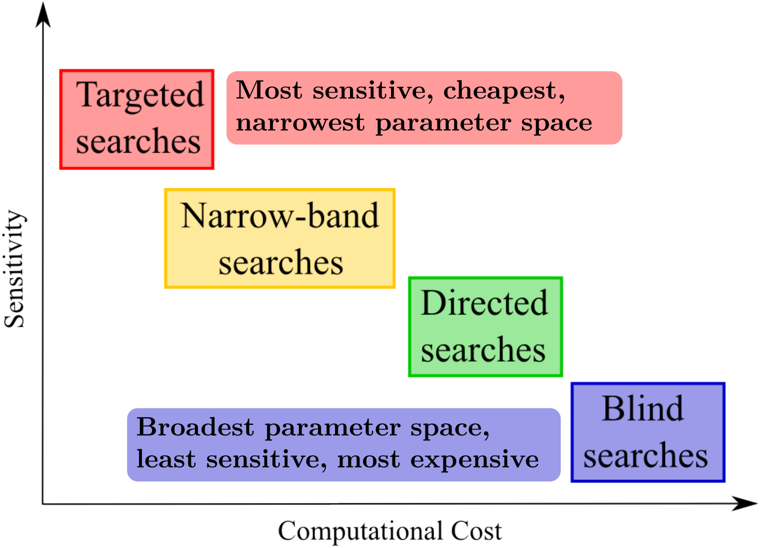
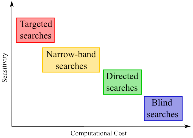


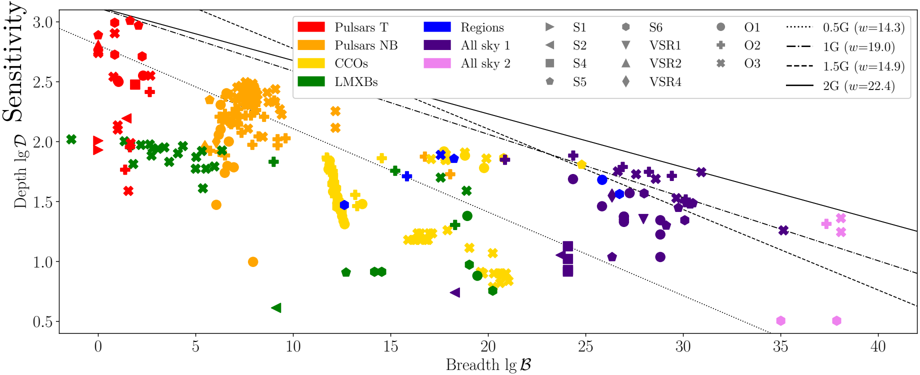
Image credits: M. Sieniawska and M. Bejger, Universe **5**, 217 (2019)

Sensitivity vs Parameter Space/Computational Cost

297 CW searches from 80 published articles (2003–2022)



Sensitivity



Parameter Space/Computational Cost

Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023), M. Sieniawska and M. Bejger, Universe **5**, 217 (2019)

Sensitivity–Parameter Space Volume

Weighted according to trend during 4 eras:

- 0.5G: pre-initial LIGO (S1–4)
- 1G: initial LIGO (S5), Virgo (VSR1)
- 1.5G: enhanced LIGO (S5), Virgo (VSR2–4)
- 2G: advanced LIGO/Virgo (O1+); see David's talk for LVK O4 plans

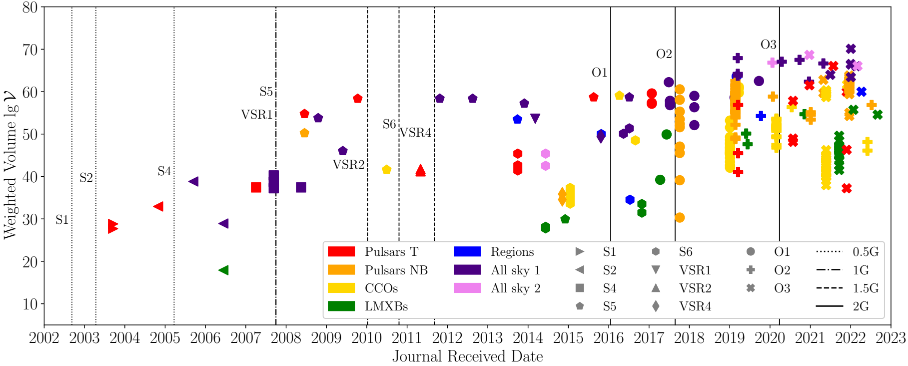


Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

Sensitivity–Parameter Space Volume

Weighted according to trend during 4 eras:

- 0.5G: pre-initial LIGO (S1–4)
- 1G: initial LIGO (S5), Virgo (VSR1)
- 1.5G: enhanced LIGO (S5), Virgo (VSR2–4)
- 2G: advanced LIGO/Virgo (O1+); see David's talk for LVK O4 plans

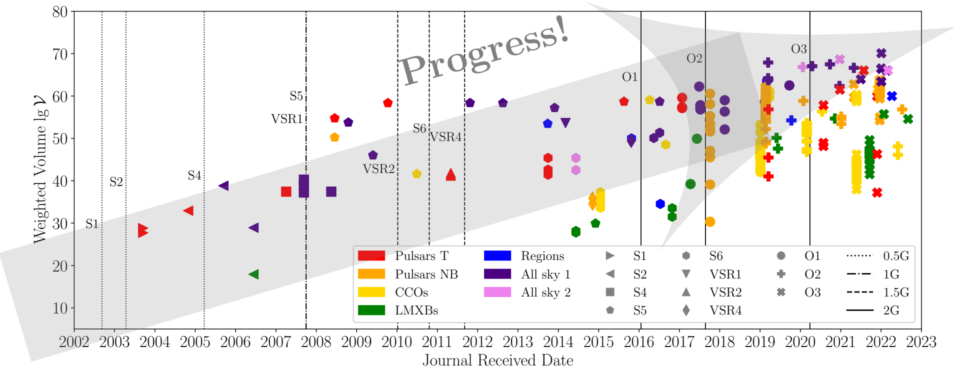
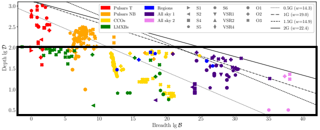
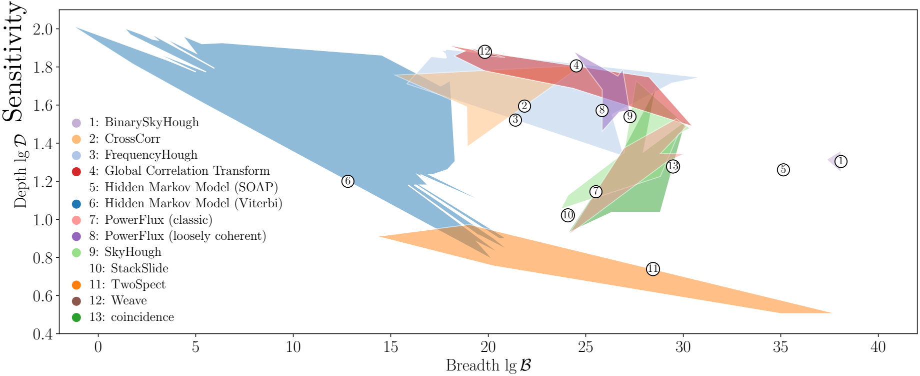


Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

Distribution of Algorithms



Sensitivity



Parameter Space/Computational Cost

Image credits: K. Wette, Astroparticle Physics **153**, 102880 (2023)

Continuous Wave Software and Practical Examples

LALSuite and CW Signal Examples

LIGO-Virgo-KAGRA **A**lgorithm **L**ibrary **Suite** (LALSuite)[29]:

- Large collection of libraries for GW data analysis, including:
  - LAL: core routines
  - LALPulsar: routines and applications for CW data analysis
- Pros:
  - Mature, packaged product: install using Pip/Conda
  - Accessible from Python through SWIGLAL interface[30]
  - API-level documentation[31] is generally pretty good
- Cons:
  - Written in C, hard to hack to do new things
  - Tutorial-level documentation largely absent

CW Signal Examples:

- Generate a traditional CW signal
- Generate a CW signal with a user-defined frequency evolution
- Link to IPython notebook in Google Colab

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[29] LIGO Scientific Collaboration et al., Free software (GPL), 2018, `https://doi.org/10.7935/GT1W-FZ16`

[30] K. Wette, SoftwareX **12**, 100634 (2020) [31] `https://lscsoft.docs.ligo.org/lalsuite/`