The problem with the *r*-modes of neutron stars

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The *r*-modes

- The *r*-modes are a family of oscillations that arise due to stellar rotation.
- They couple weakly to density perturbations and have (leading-order) frequency

$$\omega_0 = \frac{2m}{l(l+1)}\Omega.$$

- An $(l \geq 2)$ *r*-mode travels *retrograde* to the rotation in the co-rotating frame, but *prograde* in the inertial frame.

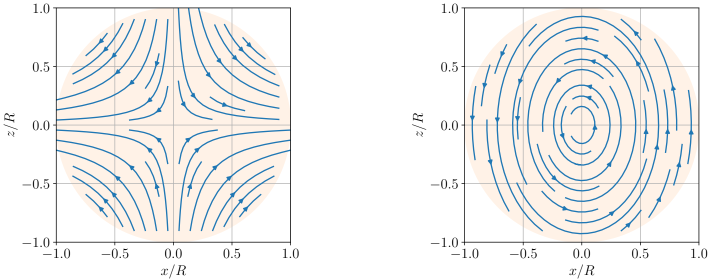

Figure 1: Streamlines of the $l = 2$ *r*-modes with $m = 1$ (left panel) and $m = 2$ (right panel) in the co-rotating frame.

Instability

- The oscillations are generically unstable to gravitational radiation (Andersson 1998; Friedman and Morsink 1998).
- Real neutron stars possess bulk and shear viscosities that *compete* with the growth of the instability. For some combination of the star's temperature and spin, the instability wins out.

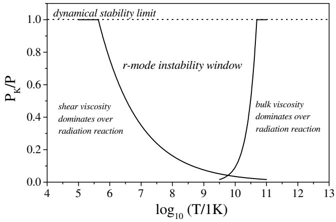

Figure 2: The *r*-mode instability window (Andersson and Kokkotas 2001).

- The instability may limit the rotation rates of young pulsars (Lindblom, Owen and Morsink 1998) and mature accreting systems (Bildsten 1998).
- Furthermore, a recent search has been conducted by the LIGO-Virgo-KAGRA Collaboration on the glitching PSR J0537-6910 (Abbott et al. 2021).

Underlying assumptions

- However, our present understanding of the *r*-modes relies on (i) Newtonian gravity and (ii) simple matter models.
- The situation is unsolved in relativity, since we (surprisingly) find a singular-eigenvalue problem (Kojima 1998):

$$(\varpi - \mu)\left[\frac{1}{jr^4}\frac{d}{dr}\left(jr^4\frac{d\Phi}{dr}\right) - v\Phi\right] = q\Phi.$$

	Newtonian theory	**general relativity**
barotropic star	One solution for each $l = \|m\|$	No solutions
non-barotropic star	Infinite solutions for each $l \geq \|m\|$	Continuous spectrum for each $l \geq \|m\|$

- This suggests that there is an issue with our assumptions.
- These problems need to be addressed to inform gravitational-wave searches and reliably extract neutron-star parameters from the signal.
- In this work, we extend the *r*-mode calculations to realistic nuclear-matter models.

Realistic nuclear matter: thermodynamics

- Mature neutron stars are cold and stratified by composition gradients,

$$d\varepsilon = \frac{\varepsilon + p}{n_{\mathsf{b}}} dn_{\mathsf{b}} + n_{\mathsf{b}} \mu_{\Delta} dY_{\mathsf{e}} \quad \Longrightarrow \quad \varepsilon = \varepsilon(n_{\mathsf{b}}, Y_{\mathsf{e}}).$$

- When the star is in β equilibrium, $\mu_{\Delta} = \mu_{\mathsf{p}} + \mu_{\mathsf{e}} - \mu_{\mathsf{n}} = 0$, the matter is barotropic, $\varepsilon = \varepsilon(n_{\mathsf{b}})$, and possesses

$$\Gamma = \frac{d \ln p}{d \ln n_{\mathsf{b}}}.$$

- Deviations from equilibrium are encoded in the *adiabatic index*

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln n_{\mathsf{b}}} \right)_{Y_{\mathsf{e}}}.$$

Simple matter models

Common assumptions in the literature include (i) $\Gamma_1 = \Gamma$ (neglecting nuclear reactions) and (ii) $\Gamma_1 = \text{const} \neq \Gamma$.

Realistic nuclear matter: stratification

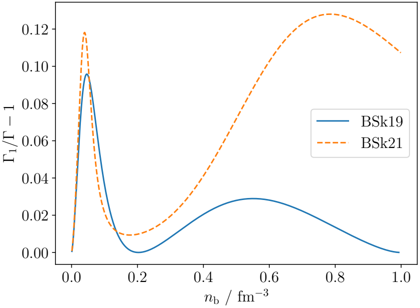

Figure 3: Stratification in nuclear-matter equations of state BSk19 and BSk21.

- A real neutron star is expected to be predominantly non-barotropic, with locally barotropic layers. Thus, $\Gamma_1 \neq$ const.

Opportunity

Probing the dense nuclear-matter equation of state is one of the main science goals of gravitational-wave astronomy.

Impact of stratification

- Consider a barotropic ($\Gamma_1 = \Gamma$) star.
  - When it is non-rotating, there exists time-independent solutions that do not source density perturbations.
  - These are zero-frequency (*polar*) *g*-modes—that have no stratification support—and (*axial*) *r*-modes.
  - As the star spins up, these solutions oscillate and mix, forming inertial modes.
  - For each $l = |m|$, one purely axial *r*-mode persists.
- Next, focus on a non-barotropic ($\Gamma_1 \neq \Gamma$) stellar model.
  - In the absence of rotation, the *g*-modes arise due to composition gradients.
  - The zero-frequency perturbations are all *r*-modes.
  - Therefore, as the star rotates, the only inertial modes are the *r*-modes.
  - The star supports the full set of $l \geq |m|$ *r*-modes, including radial overtones.
- However, neither of these two regimes describe a neutron star.

Globally non-barotropic stars

- The r-mode frequency is analytic at leading order in $\epsilon = \Omega/\sqrt{GM/R^3}$. The solution is obtained by solving the eigenvalue problem for $\tilde{\omega}_2$, where

$$\omega = \omega_0 + \omega_2 + \mathcal{O}(\epsilon^5)$$

  and $\tilde{\omega}_2 = \omega_2/\sqrt{GM/R^3}$.
- We explore what happens to the solutions as $\Gamma_1 \to \Gamma$: except for the fundamental $l = |m|$ solution, all the perturbations diverge and become generic inertial modes.
- Note that their frequencies may no longer satisfy the instability criterion.

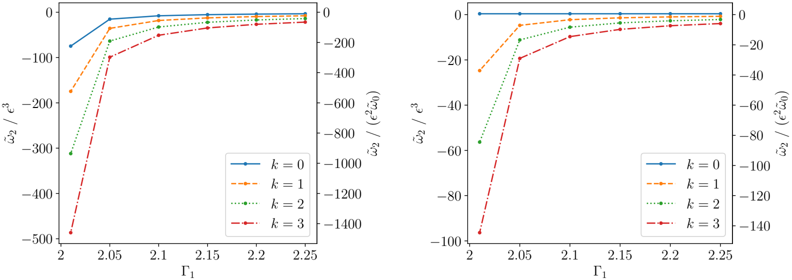

Figure 4: Eigenfrequencies of the $l = 2$ r-modes with $m = 1$ (left panel) and $m = 2$ (right panel) as $\Gamma_1 \to \Gamma = 2$.

Realistic nuclear matter: eigenfrequencies

- We develop the r-mode perturbation equations to allow for realistic neutron-star matter with local barotropicity. To do this, we focus on $l = |m|$ solutions.

Table 1: Eigenfrequencies $\tilde{\omega}_2/\epsilon^3$ of the $l = |m|$ r-modes.

	$m=3$		$m=2$		$m=1$	
k	BSk19	BSk21	BSk19	BSk21	BSk19	BSk21
0	0.4278	0.4282	0.3986	0.3989	0.0000	0.0000
1	-2.6945	-0.4610	-6.2134	-1.2297	-18.2599	-3.7280
2	-5.1463	-1.6301	-14.6769	-3.9823	-53.6627	-11.9555
3	-11.5551	-3.1305	-28.7989	-7.6752	-94.0447	-23.5897

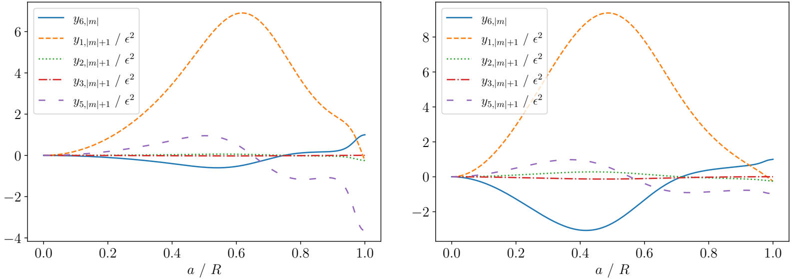

Figure 5: Eigenfunctions of the $(l, m, k) = (|m|, 3, 1)$ r-modes for BSk19 (left panel) and BSk21 (right panel).

Reformulating the problem

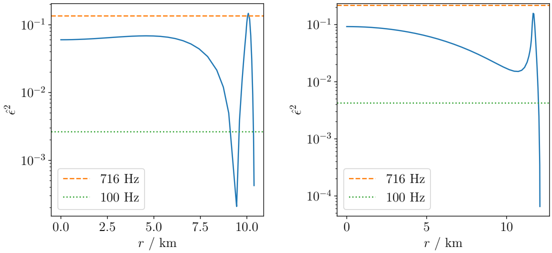

Figure 6: Dimensionless Brunt-Väisälä frequency for BSk19 (left panel) and BSk21 (right panel).

- For fast-spinning neutron stars, we argue that the stratification is weak and the problem resembles the inertial modes, which has been solved in relativity (Idrisy, Owen and Jones 2015). (The next step would be to solve the inertial modes with realistic stratification.)
- For slower spins, the problem will need to be solved numerically from the outset.

Conclusions

- The r-modes with their associated gravitational-wave instability are a promising source for continuous-wave detection.
- However, we are unable to determine the r-modes in general relativity.
- We studied the effects of matter on the spectrum and found that
  - (i) as the star changes from being globally non-barotropic to being barotropic, all but the fundamental $l = |m|$ r-mode diverge and join the inertial-mode family; and
  - (ii) for realistic nuclear equations of state, the presence of locally barotropic regions results in the same behaviour.
- Because of these effects, we suggest reformulations of the problem to calculate the inertial-mode spectrum with realistic nuclear matter.

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