The weak-signal Bayes-factor approximation

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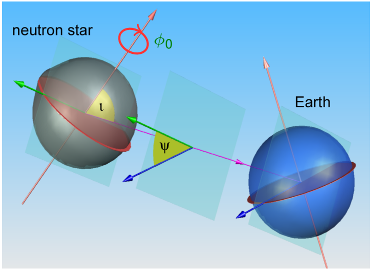
Signal amplitude parameters



☞ Amplitude parameters:

$$\mathcal{A} = \{h_0, \underbrace{\cos\iota, \psi, \phi_0}\}$$

"radius" ($h_0 \geq 0$) + 3 Euler angles

$\mathcal{A}$ *generally* unknown

Any spin orientation equally likely (isotropic):

$$P\left(\cos\iota | I\right) = \mathrm{unif}\left(-1, 1\right)$$
$$P\left(\psi | I\right) = \mathrm{unif}\left(0, 2\pi\right)$$

Any initial phase equally likely:

$$P\left(\phi_0 | I\right) = \mathrm{unif}\left(0, 2\pi\right)$$

☞ "physical prior" $P\left(h_0, \cos\iota, \psi, \phi_0 | I\right) = \mathrm{const} \times \underbrace{P\left(h_0 | I\right)}_{?}$

Part I: Pre-history

From $\mathcal{F}(x)$- to $\mathcal{B}(x)$-statistic

JKS coordinates $\{\mathcal{A}^\mu\}$

[Jaranowski,Królak,Schutz PRD58(1998)]

"cartesian" JKS coordinates $\mathcal{A}^\mu = \mathcal{A}^\mu(h_0, \cos\iota, \psi, \phi_0)$

$$\begin{aligned}
\mathcal{A}^1 &= \ \ 0.5 h_0 (1+\cos^2\iota)\cos 2\psi\ \cos\phi_0 - h_0\cos\iota\ \sin 2\psi\ \sin\phi_0 \\
\mathcal{A}^2 &= \ \ 0.5 h_0 (1+\cos^2\iota)\sin 2\psi\ \cos\phi_0 + h_0\cos\iota\ \cos 2\psi\ \sin\phi_0 \\
\mathcal{A}^3 &= -0.5 h_0 (1+\cos^2\iota)\cos 2\psi\ \sin\phi_0 - h_0\cos\iota\ \sin 2\psi\ \cos\phi_0 \\
\mathcal{A}^4 &= -0.5 h_0 (1+\cos^2\iota)\sin 2\psi\ \sin\phi_0 + h_0\cos\iota\ \cos 2\psi\ \cos\phi_0
\end{aligned}$$

☞ Signal strain @ detector

$$h(t;\mathcal{A};\lambda) = \sum_{\mu=1}^{4} \mathcal{A}^\mu\, h_\mu(t;\lambda)$$

$$h_1(t;\lambda) = a(t;\vec{n}_{\rm sky})\ \cos\phi(t;\lambda), \quad h_2(t;\lambda) = b(t;\vec{n}_{\rm sky})\ \cos\phi(t;\lambda)$$
$$h_3(t;\lambda) = a(t;\vec{n}_{\rm sky})\ \sin\phi(t;\lambda), \quad h_4(t;\lambda) = b(t;\vec{n}_{\rm sky})\ \sin\phi(t;\lambda)$$

Phase-evolution parameters: $\lambda = \{\vec{n}_{\rm sky}, \textit{freq}, \textit{spindown}, \textit{binary}, \ldots\}$

(here: λ assumed known)

Likelihood ratio $\mathcal{L}(x;\mathcal{A},\lambda)$

$\mathcal{H}_{\rm G}$: data = Gaussian noise: $x(t) = n(t)$
$\mathcal{H}_{\rm S}$: data = Gaussian noise + signal: $x(t) = n(t) + h(t;\mathcal{A},\lambda)$

Goal: given $x(t)$ ☞ "optimally" decide between $\mathcal{H}_{\rm G}$ and $\mathcal{H}_{\rm S}$

☞ likelihood ratio: Gaussian in $\mathcal{A}^\mu$

$$\mathcal{L}(x;\mathcal{A}) \equiv \frac{P\left(x|\mathcal{H}_{\rm S},\mathcal{A}\right)}{P\left(x|\mathcal{H}_{\rm G}\right)} = \exp\left[-\frac{1}{2}\mathcal{A}^\mu \mathcal{M}_{\mu\nu}\mathcal{A}^\nu + \mathcal{A}^\mu x_\mu\right]$$

4 matched-filter outputs $x_\mu(\lambda) \equiv (x|h_\mu(\lambda)) \in \mathbb{R}$
4x4 antenna-pattern matrix:

$$\mathcal{M}_{\mu\nu}(\vec{n}_{\rm sky}) \equiv (h_\mu|h_\nu) = \underbrace{(S_{\rm n}^{-1}T)}_{\gamma\equiv\text{"quality x quantity"}} \begin{pmatrix} A & C & & \\ C & B & \multicolumn{2}{c}{O} \\ & & A & C \\ \multicolumn{2}{c}{O} & C & B \end{pmatrix}$$

matched-filter scalar product: $(x|y) \equiv 2S_{\rm n}^{-1}\int x(t)\,y(t)dt = 2\gamma\,\langle xy\rangle$

$A \equiv \left\langle a^2\right\rangle$, $B \equiv \left\langle b^2\right\rangle$, $C \equiv \langle ab\rangle$

1998: Maximum-likelihood: $\mathcal{F}$-statistic

☞ analytically maximize $\mathcal{L}$ over $\mathcal{A}^\mu$: [Jaranowski,Królak,Schutz PRD58(1998)]

$$\mathcal{F}(x) \equiv \max_{\{\mathcal{A}^\mu\}} \ln \mathcal{L}(x; \mathcal{A}^\mu) = \frac{1}{2} x_\mu \left(\mathcal{M}^{-1}\right)^{\mu\nu} x_\nu$$

$$2\mathcal{F}(x) = 2D^{-1}\left[AB(\mathcal{F}_{\mathrm{A}}(x) + \mathcal{F}_{\mathrm{B}}(x)) - 2C^2 \mathcal{F}_{\mathrm{C}}(x)\right]$$

$$2\mathcal{F}_{\mathrm{A}}(x) \equiv \frac{x_1^2 + x_3^2}{\gamma A}, \quad 2\mathcal{F}_{\mathrm{B}}(x) \equiv \frac{x_2^2 + x_4^2}{\gamma B}, \quad 2\mathcal{F}_{\mathrm{C}}(x) \equiv \frac{x_1 x_2 + x_3 x_4}{\gamma C}$$

sub-determinant $D = \det \begin{pmatrix} A & C \\ C & B \end{pmatrix} = AB - C^2$

☞ Long thought to be “optimal”, until . . .

2009: $\mathcal{B}$-statistic [Prix,Krishnan CQG26(2009)]

Neyman-Pearson-Searle theorem [Searle arXiv:0804.1161]

☞ Bayes factor is the optimal statistic (if drawing from prior)

$$B_{\mathrm{S/G}}(x) \equiv \frac{P\,(x|\mathcal{H}_{\mathrm{S}})}{P\,(x|\mathcal{H}_{\mathrm{G}})} = \int \mathcal{L}(x;\mathcal{A})\,P\,(\mathcal{A}|I)\;d^4\mathcal{A}$$

- "$\mathcal{F}$-statistic prior" $P\,(\mathcal{A}^\mu) = \mathrm{const}$ ☞ $B_{\mathrm{S/G}} \propto e^{\mathcal{F}}$

  (but prior is unphysical: $P\,(h_0, \cos\iota, \psi, \phi_0) \propto {h_0}^3(1-\cos^2\iota)^3$)

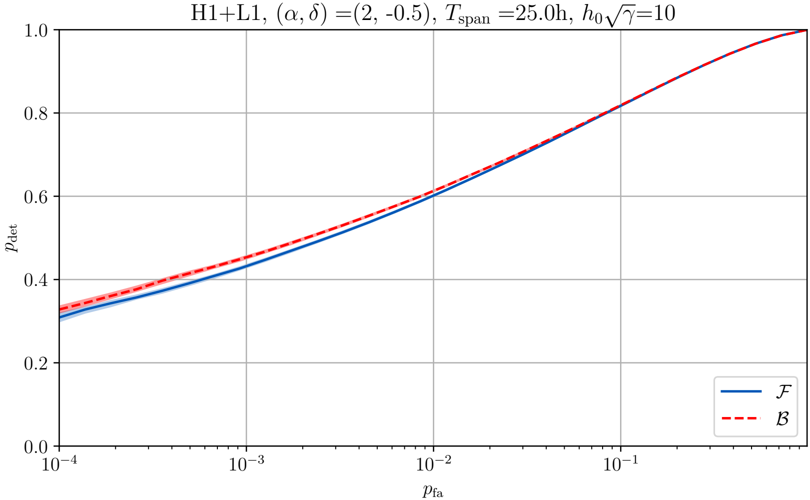
- "physical prior" $P\,(h_0, \cos\iota, \psi, \phi_0) = \mathrm{const} \times P\,(h_0)$
  - ☞ can integrate ϕ_0 ✓
  - ☞ if $P\,(h_0) = \mathrm{const} \Rightarrow$ can integrate h_0! ✓

☞ "$\mathcal{B}$-statistic": 2D integral [Prix,Krishnan CQG26(2009)],[Whelan+ CQG31(2014)]

$$\mathcal{B}(x) \propto \int \frac{1}{\varrho}\,e^{\Theta}\,I_0(\Theta)\;d\cos\iota\;d\psi$$

$$\Theta(x;\cos\iota,\psi) \equiv \frac{q^2}{4\varrho^2},\, q = q(x;\cos\iota,\psi),\, \varrho = \varrho(\cos\iota,\psi)$$

$\mathcal{B}(x)$ beats $\mathcal{F}(x)$



various attempts to solve/approximate $\mathcal{B}(x)$: [Dergachev PRD85(2012)][Whelan+ CQG31(2014)][Haris,Pai PRD96(2017)][Dhurandhar+ arXiv:1707.08163][Bero,Whelan CQG36(2019)][Wette Universe7(2021)]

Part II: Recent surprises

Short segments and segment weights

Surprise 1: dominant-response $\mathcal{F}_{\mathrm{AB}}$-statistic [Covas,Prix PRD105(2022)]

Fully Coherent Search

Coherent + Coherent + Coherent + Coherent + Coherent

Sum = Semicoherent Search

All-sky binary search ☞ very short segments $T_{\mathrm{seg}} \sim \mathcal{O}(100s)$

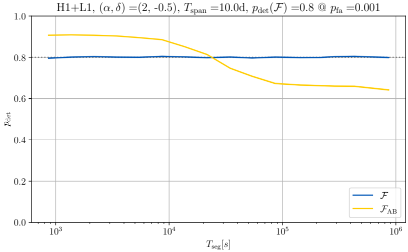
☞ degenerate $\mathcal{F}(x)$ in certain $\vec{n}_{\mathrm{sky}}$ where $D \to 0$

☞ maximum-likelihood $\mathcal{A}^{\mu}{}_{\mathrm{ML}} = \mathcal{M}^{-1\,\mu\nu} x_{\mu}$ fails!

Fix: "dominant-response" statistic:

$$\mathcal{F}_{\mathrm{AB}}(x) \equiv \begin{cases} \mathcal{F}_{\mathrm{A}}(x) & \text{if } A > B \\ \mathcal{F}_{\mathrm{B}}(x) & \text{otherwise} \end{cases}$$

☞ beats $\mathcal{F}(x)$ for short segments by up to $\sim 19\%$!!

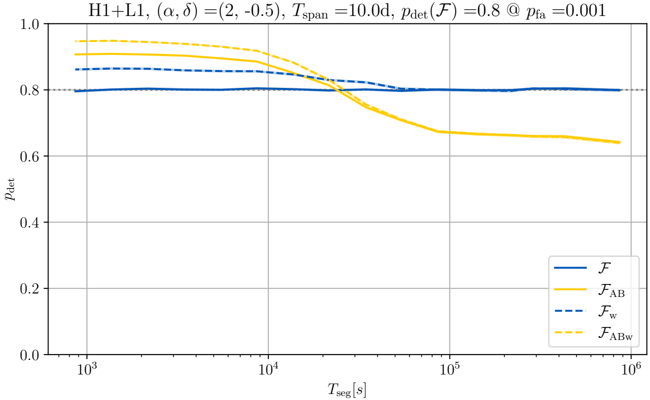


Surprise 2: segment weighting [Covas,Prix PRD106(2022)]

Can improve statistics by (empirical) *per-segment weights* w_ℓ:

("data factor" $\gamma = S_{\mathrm{n}}^{-1} T$)

- $\widehat{\mathcal{F}}_w \propto \sum_{\ell=1}^{N_{\mathrm{seg}}} \left(\gamma\,(A+B)\right)_\ell\,\mathcal{F}_\ell$
- $\widehat{\mathcal{F}}_{\mathrm{ABw}} \propto \sum_{\ell=1}^{N_{\mathrm{seg}}} \left(\gamma\left(Q + \frac{C^2}{Q}\right)\right)_\ell \mathcal{F}_{\mathrm{AB},\ell}$ (with $Q = \max[A, B]$)



☞ $\mathcal{F}$-statistic far from optimal for short segments!

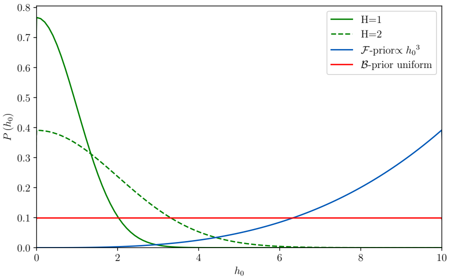
Part III

Is there a ~~better~~ Bayesian way?

New h_0-prior

Half-normal distribution with h_0-"scale" H

$$P\left(h_0 | I, H\right) = \frac{\sqrt{2}}{\sqrt{\pi}\, H}\, e^{-\frac{{h_0}^2}{2H^2}} \quad \text{for } h_0 \geq 0$$



- ✓ favors weak signals over strong
- ✓ proper prior
- ✓ $H \to \infty$ ☞ reduces to uniform-h_0
- ✓ conjugate to $\mathcal{L}$: can integrate h_0!

Generalized $\mathcal{B}$-statistic

"relative scale" $H_* \equiv H\sqrt{\gamma}$

$$\mathcal{B}(x; H_*) = \frac{1}{\pi}\int d\cos\iota\, d\psi \frac{1}{\sqrt{1 + H_*^2 \varrho^2}}\, e^{\Theta}\, I_0(\Theta)$$

$$\Theta(x; \cos\iota, \psi) \equiv \frac{H_*^2\, q^2(x; \cos\iota, \psi)}{4(1 + H_*^2\, \varrho^2(\cos\iota, \psi))}$$

Two interesting limits:

- strong-signal limit: $H_* \gg 1$ ☞ recover const-h_0 $\mathcal{B}(x)$-stat
- weak-signal limit: $H_* \ll 1$: Taylor-expand in H_*!

(coherent sensitivity estimate $h_0|_{p_{\mathrm{fa}}=1\%}^{p_{\mathrm{det}}=90\%} \sim 11.4/\sqrt{\gamma}$ ☞ $h_0^{90\%}\sqrt{\gamma} \sim \mathcal{O}(10) \not\ll 1$)

Analytic weak-signal limit $H_* \ll 1$

Taylor-expand $\mathcal{B}(x; H_*)$ in $\mathcal{O}(H_*)$:

$$\mathcal{B}(x) = 1 + \frac{H_*^2}{4} \left\langle q^2 - 2\varrho^2 \right\rangle_{\cos\iota,\psi} + \mathcal{O}\left(H_*^4\right)$$

☞ **Weak-signal limit Bayes factor $\mathcal{B}_{\mathrm{weak}}(x)$**

$$\ln \mathcal{B}_{\mathrm{weak}}(x) \equiv \frac{H^2}{10} \gamma \left[A\left(2\mathcal{F}_{\mathrm{A}}(x) - 2\right) + B\left(2\mathcal{F}_{\mathrm{B}}(x) - 2\right) \right]$$

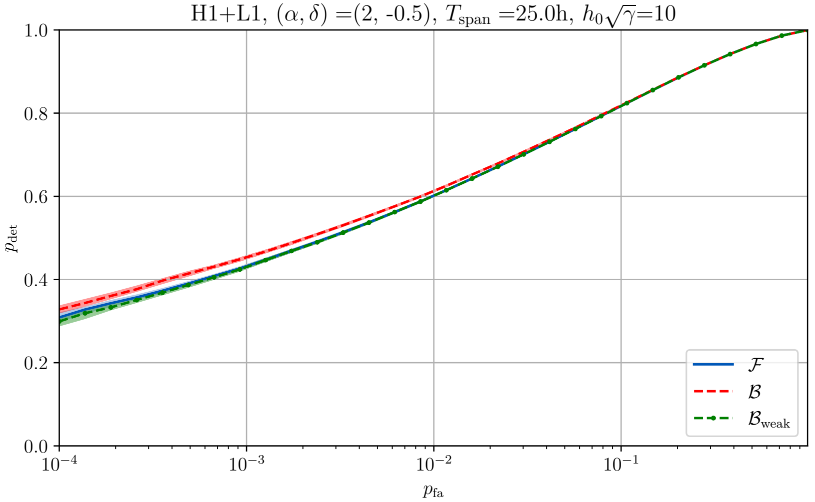
☞ generalized χ^2-distribution with (2,2) degrees of freedom

☞ already includes segment weights $(\gamma A, \gamma B)_\ell$?

Compare:

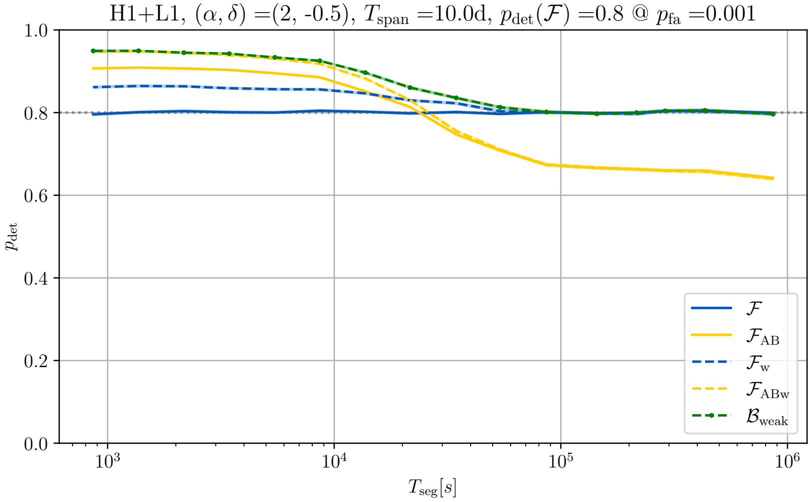
- $2\mathcal{F}(x) = 2D^{-1}\left[AB(\mathcal{F}_{\mathrm{A}}(x) + \mathcal{F}_{\mathrm{B}}(x)) - 2C^2\mathcal{F}_{\mathrm{C}}(x)\right] \sim \chi^2_{4\mathrm{dof}}$
- short-segment $\mathcal{F}_{\mathrm{AB}}(x) = \begin{cases} \mathcal{F}_{\mathrm{A}}(x) & \text{if } A > B \\ \mathcal{F}_{\mathrm{B}}(x) & \text{otherwise} \end{cases} \sim \chi^2_{2\mathrm{dof}}$

Results

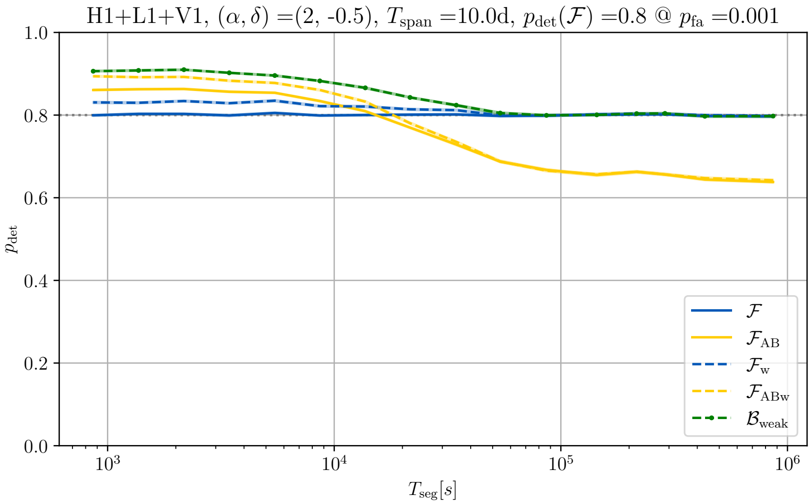


H1+L1, (α, δ) =(2, -0.5), T_{span} =25.0h, $h_0\sqrt{\gamma}$=10
1.0
0.8
0.6
0.4
0.2
0.0
p_{det}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
p_{fa}
$\mathcal{F}$
$\mathcal{B}$
$\mathcal{B}_{\mathrm{weak}}$

Results (H1+L1)



Results (H1+L1+V1)



Summary

- Half-normal h_0-prior generalizes $\mathcal{B}$-statistic ☞ $\mathcal{B}(x; H)$
- "weak-signal" limit $H \ll 1/\sqrt{\gamma}$ ☞ new analytical solution!
- for longer segments: similar to $\mathcal{F}_{\mathrm{w}}$, weaker than $\mathcal{B}$
- for short segments $\lesssim$ 1d: equal or better than $\mathcal{F}_{\mathrm{ABw}}$
- $\ln \mathcal{B}_{\mathrm{weak}} = \frac{H^2}{10}(\underbrace{\sum_\mu x_\mu^2} - \mathrm{Tr}\mathcal{M})$ vs $2\mathcal{F} = x_\mu (\mathcal{M}^{-1})^{\mu\nu} x_\nu$

  ☞ "5-vector statistic" [Astone+ CQG27(2010)],[D'Onofrio+ arXiv:2406.09236v]
- future work: better line-robust statistics $B_{\mathrm{S/GL}}(H)$ based on proper h_0-prior?