Bayesian $\mathcal{F}$-statistic-based parameter estimation of continuous gravitational waves from known pulsars

Based on A. Ashok et al., PRD 109, 104002 (2024)

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Outline

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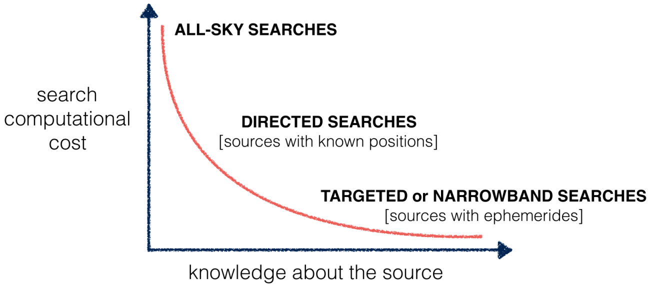
Targeted CW searches

Types of CWs searches using data from ground-based detectors like Advanced LIGO:



- Massive reduction in parameter-space size compared to wide parameter-space searches: a targeted search can use a fully coherent combination of all the data, **leading to the maximum possible sensitivity**
- Presence of a neutron star is assured and its rotational frequency and spin-down are known: **a null measurement is directly informative about the gravitational-wave emission of the source**

Motivation for a new pipeline

- We propose a new pipeline to carry out targeted searches
- Previously, **only a single Bayesian method existed** for parameter estimation on known pulsars, often referred to as the Time Domain method or Heterodyne method
- We introduce a new expression and implementation of the CW signal likelihood function, based on the **well-established and efficient** $\mathcal{F}$-statistic framework
- Instead of analytically maximizing the likelihood ratio over the 4 amplitude parameters, we set up priors on them and keep the complete likelihood or analytically marginalize over ϕ_0
- This approach can be used for any combination of parameters describing the CW signal, but here we focus on targeted searches, **only computing the posterior over the unknown amplitude parameters**

CW signal model

- Varying mass quadrupole moment in a fast-rotating neutron star. In the absence of precession, the **signal is expected at twice the rotation frequency** and with twice the rotational spin-down of the star.
- The signal strain s in the detector X has the form

$$s^X(t) = F_+(t;\alpha,\delta,\psi)\, h_+(t) + F_\times(t;\alpha,\delta,\psi)\, h_\times(t),$$

where "$+$" and "$\times$" indicate the two gravitational-wave polarizations, and $F_+(t;\alpha,\delta,\psi)$ and $F_\times(t;\alpha,\delta,\psi)$ are the detector antenna-pattern functions. The two waveforms $h_+(t)$ and $h_\times(t)$ are given by

$$h_+(t) = \frac{h_0}{2}(1+\cos^2\iota)\cos[\phi(t)+\phi_0],$$
$$h_\times(t) = h_0 \cos\iota \sin[\phi(t)+\phi_0].$$

Amplitude and phase-evolution parameters

Two different types of **parameters** define the CW signal:

Amplitude A

- Gravitational-wave amplitude: $h_0 = C \frac{I_{zz} \epsilon f_0^2}{d}$
- Inclination angle: ι
- Polarization angle: ψ
- Initial phase: ϕ_0

Phase-evolution λ

- Frequency evolution: f_0, f_1, ...
- Sky position: α and δ
- Keplerian binary orbit: $a_{\rm p}$ (projected semi-major axis), $P_{\rm orb}$ (orbital period), $t_{\rm asc}$ (time of ascension), e (eccentricity), and ω (argument of periapsis)

In a targeted search, **all phase-evolution parameters are known** or uncertainty is small enough to allow for a fully coherent search

Signal model in JKS amplitude parameters

CW **signal depends non-linearly on the physical amplitude parameters** A. A different set of amplitude parameters $\mathcal{A}$ linearize the CW signal $s^X(t)$ in the frame of detector X:

$$s^X(t;\mathcal{A},\lambda) = \sum_{\mu=1}^{4} \mathcal{A}^\mu \, h_\mu^X(t;\lambda),$$

where h_μ^X are basis functions (depend on phase-evolution parameters) and

$$\mathcal{A}^1 \equiv \quad \frac{h_0(1+\cos^2\iota)}{2} \cos\phi \, \cos\psi - h_0 \cos\iota \, \sin\phi \, \sin\psi \,,$$

$$\mathcal{A}^2 \equiv \quad \frac{h_0(1+\cos^2\iota)}{2} \cos\phi \, \sin\psi + h_0 \cos\iota \, \sin\phi \, \cos\psi \,,$$

$$\mathcal{A}^3 \equiv - \frac{h_0(1+\cos^2\iota)}{2} \sin\phi \, \cos\psi - h_0 \cos\iota \, \cos\phi \, \sin\psi \,,$$

$$\mathcal{A}^4 \equiv - \frac{h_0(1+\cos^2\iota)}{2} \sin\phi \, \sin\psi + h_0 \cos\iota \, \cos\phi \, \cos\psi$$

Outline

Likelihood ratio I

We define the scalar product:

$$(x|y) \equiv 2 \sum_{X}^{N_{\rm det}} S_X^{-1} \int_0^T x^X(t)\, y^X(t)\, dt$$

- The noise hypothesis $\mathcal{H}_{\rm N}$ states that the strain data $x^X(t)$ contains only (Gaussian) noise $n^X(t)$, i.e., $s^X = 0$. It can be shown that the likelihood function for $\mathcal{H}_{\rm N}$ is:

$$P\,(x|\mathcal{H}_{\rm N}) = \kappa\, e^{-\frac{1}{2}(x|x)}$$

- The signal hypothesis $\mathcal{H}_{\rm S}$ states that the strain data $x^X(t)$ contains a signal $s^X(t)$ in addition to (Gaussian) noise $n^X(t)$:

$$x^X(t) = n^X(t) + s^X(t; \mathcal{A}, \lambda)$$

$$P\,(x|\mathcal{H}_{\rm S}, \mathcal{A}, \lambda) = \kappa\, e^{-\frac{1}{2}(x-s|x-s)}$$

Likelihood ratio II

To decide whether the signal or noise hypothesis is favored by the data x, both frequentist and Bayesian frameworks **require expressing the likelihood ratio between the two hypotheses** $\mathcal{L}$:

$$\mathcal{L}(x;\mathcal{A},\lambda) \equiv \frac{P\left(x|\mathcal{H}_{\mathrm{S}},\mathcal{A},\lambda\right)}{P\left(x|\mathcal{H}_{\mathrm{N}}\right)} = e^{(x|s)-\frac{1}{2}(s|s)}\,,$$

which when substituting the signal $s^X = \sum_{\mu=1}^{4} \mathcal{A}^\mu\, h_\mu^X(t;\lambda)$ becomes:

$$\log\mathcal{L}(x;\mathcal{A},\lambda) = \mathcal{A}^\mu x_\mu - \frac{1}{2}\mathcal{A}^\mu \mathcal{M}_{\mu\nu}\mathcal{A}^\nu = \mathcal{A}^\mu x_\mu - \frac{\rho^2}{2}\,,$$

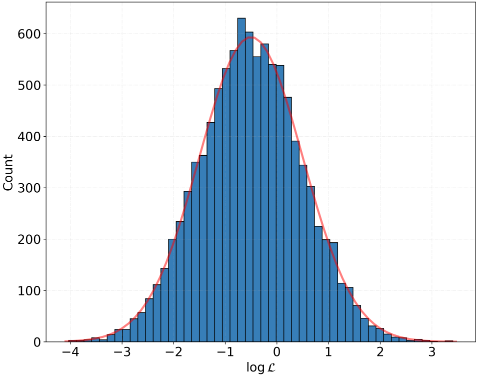
with implicit summation over repeated amplitude indices $\mu,\nu = 1\ldots 4$, and the definitions

$$x_\mu(\lambda) \equiv (x|h_\mu) \quad \text{and} \quad \mathcal{M}_{\mu\nu}(\lambda) \equiv (h_\mu|h_\nu)$$

This likelihood is **mathematically equivalent to the one used by the Time-domain heterodyning method**, although its calculation is done differently

Statistical properties

$\log \mathcal{L}$ is a **Gaussian-distributed quantity** since each x_μ follows a Gaussian distribution.



Histogram of 10 000 log-likelihood ratio values for the noise-only case. The red line shows the expected Gaussian distribution with a mean of -0.5 and a standard deviation of 1.

Marginalized likelihood I

The likelihood ratio $\mathcal{L}$ **can be analytically marginalized over** ϕ_0. Some advantages:

- It leaves us with less dimensions to explore numerically (4 to 3)
- It avoids the bi-modality of posteriors in $\psi - \phi_0$
- It provides a consistency check for the results from the full-likelihood

We can **explicitly factor out the** ϕ_0**-dependence** in the $\mathcal{A}^\mu x_\mu$ term:

$$\mathcal{A}^\mu x_\mu = q \cos(\phi_0 - \varphi_0),$$

where q and φ_0 are independent of ϕ_0.

Marginalized likelihood II

We can see that the signal power ρ^2 does not depend on ϕ_0, and therefore writing the likelihood ratio in the form

$$\mathcal{L}(x;\mathcal{A}) = e^{-\frac{1}{2}\rho^2} e^{q\cos(\phi_0-\varphi_0)}$$

makes the ϕ_0 dependence fully explicit. Using a uniform ϕ_0-prior we can **obtain the ϕ_0-marginalized likelihood ratio** $\mathcal{L}^{\overline{\phi_0}}$ in the form

$$\begin{aligned}
\mathcal{L}^{\overline{\phi_0}} &\equiv \int_0^{2\pi} \mathcal{L}(x;\mathcal{A})\, P\left(\phi_0 | \mathcal{H}_{\mathrm{S}}\right)\, d\phi_0 \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}(x;\mathcal{A})\, d\phi_0 \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}\rho^2} \int_0^{2\pi} e^{q\cos(\phi_0-\varphi_0)}\, d\phi_0 = e^{-\frac{1}{2}\rho^2} I_0(q),
\end{aligned}$$

where we used the Jacobi-Anger expansion to see that $\int_0^{2\pi} e^{q\,\cos\phi}\, d\phi = 2\pi\, I_0(q)$ in terms of the modified Bessel function of the first kind I_0.

Priors on amplitude parameters I

If there are **no observational constraints on the rotation axis of the pulsar, we assume isotropic "ignorance" priors** on the angle parameters:

- The initial phase ϕ_0 corresponds to the pulsar rotation angle at a reference time and the ignorance prior is uniform over the range $\phi_0 \in [0, 2\pi)$
- The ignorance prior for the direction of the rotation axis is also uniform $\in [0, 2\pi)$ and it translates to uniform priors in $\cos\iota \in [-1, 1]$ and $\psi \in [0, 2\pi)$
- We see that $\psi \to \psi + \pi$ leaves the $\mathcal{A}^\mu$ unchanged, and further that $\psi \to \psi + \pi/2$ flips their sign, which can be compensated by $\phi_0 \to \phi_0 + \pi$. We can therefore choose a gauge where $\psi \in [-\pi/4, \pi/4)$ and $\phi_0 \in [0, 2\pi)$.

When pulsar observations do constrain these priors, they can be modified appropriately.

Priors on amplitude parameters II

For h_0 the choice of prior range $[h_{\rm low}, h_{\rm high}]$ and probability distribution is less straightforward:

- For a known pulsar, one could inform $h_{\rm high}$ from the observed pulsar parameters, namely the spindown upper limit $h_{\rm high} < h_0^{\rm sd}$
- If a previous search has established $h_0^{\rm UL}$ for the pulsar, then, under the assumption that the signal amplitude does not change over time, one could require $h_{\rm high} < h_0^{\rm UL}$
- Use theoretical range of ellipticities ε of neutron stars to derive an h_0 prior range from $h_0(\varepsilon) = \frac{4\pi^2 G \varepsilon I_{zz} f^2}{c^4 d}$
- Quadrupolar deformations can be sourced by magnetic field B: smallest expected signal may correspond to this, so $h_{\rm low} \sim h_0(B)$

For a strong signal, the prior has minimal influence on the resulting posterior (likelihood strongly peaked). For a weak signal, a uniform prior on h_0 leads to a more conservative upper limit compared to a log-uniform distribution. Conversely, a log-uniform prior ensures a uniform sampling when our ignorance spans several orders of magnitude.

Bayes' theorem

Using Bayes' theorem, **the posterior for the unknown amplitude parameters $\mathcal{A}$ is**:

$$P\left(\mathcal{A}|x, \mathcal{H}_{\mathrm{S}}, \lambda\right) = P\left(\mathcal{A}|\mathcal{H}_{\mathrm{S}}, \lambda\right) \frac{P\left(x|\mathcal{H}_{\mathrm{S}}, \mathcal{A}, \lambda\right)}{P\left(x|\mathcal{H}_{\mathrm{S}}, \lambda\right)}$$

$$= P\left(\mathcal{A}|\mathcal{H}_{\mathrm{S}}, \lambda\right) \frac{\mathcal{L}(x; \mathcal{A}, \lambda)\, P\left(x|\mathcal{H}_{\mathrm{N}}\right)}{P\left(x|\mathcal{H}_{\mathrm{S}}, \lambda\right)}$$

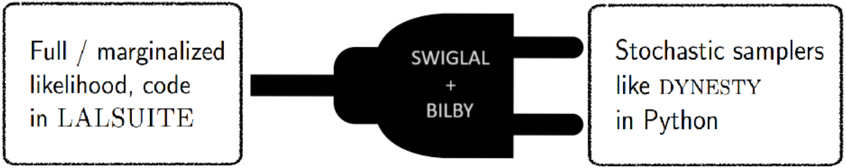
where $P\left(\mathcal{A}|\mathcal{H}_{\mathrm{S}}, \lambda\right)$ is the prior on the amplitude parameters, $P\left(x|\mathcal{H}_{\mathrm{S}}, \mathcal{A}, \lambda\right)$ is the signal likelihood, and $P\left(x|\mathcal{H}_{\mathrm{S}}, \lambda\right)$ is the amplitude-marginalized signal likelihood.

Collecting all $\mathcal{A}$-independent factors into a proportionality constant k:

$$P\left(\mathcal{A}|x, \mathcal{H}_{\mathrm{S}}, \lambda\right) = k\, \mathcal{L}(x; \mathcal{A}, \lambda)\, P\left(\mathcal{A}|\mathcal{H}_{\mathrm{S}}, \lambda\right)$$

where k can be determined via the normalization condition $\int P\left(\mathcal{A}|...\right)\, d^4\mathcal{A} = 1$.

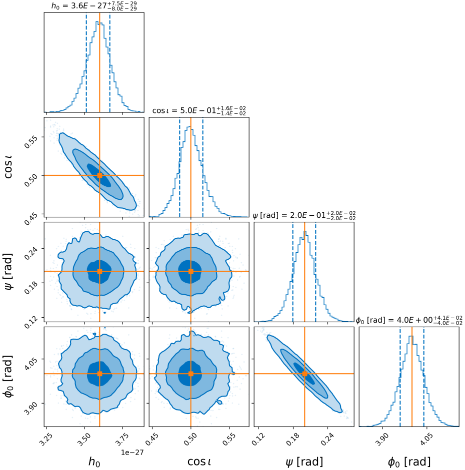
Software



- Two stochastic samplers tested: DYNESTY and BILBYMCMC
- On an Intel(R) Xeon(R) CPU E5-2620 v4 @ 2.10 GHz processor, the median time for a single computation of the likelihood function is $\sim \mathcal{O}(\mu\mathrm{s})$
- The ϕ_0-marginalized likelihood needs around twice as the full likelihood
- The known-pulsar search we did took 154 seconds (with 32-core parallelization) with the full likelihood

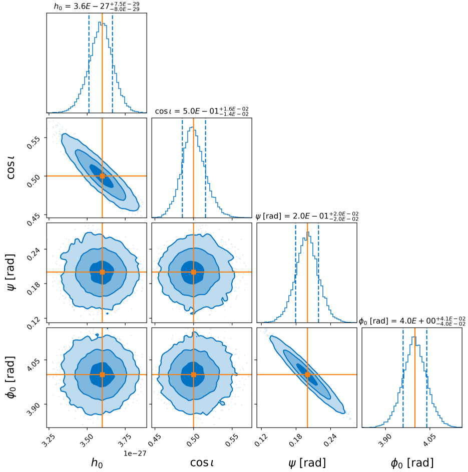
Outline

Single signal without noise:

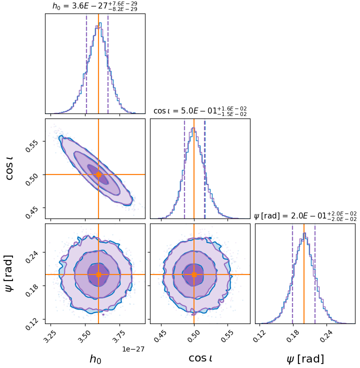


Full likelihood

Single signal without noise:



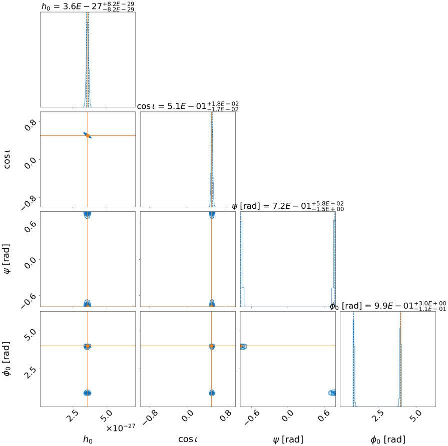
Full likelihood



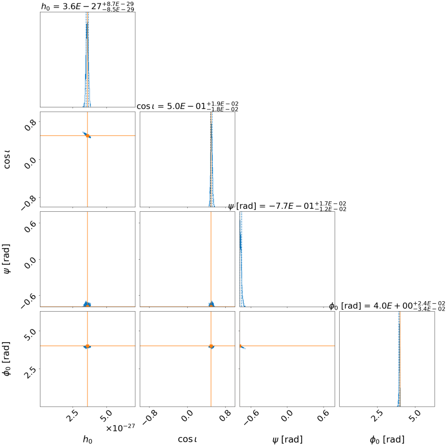
Marginalized likelihood

Tests I: Dynesty vs BilbyMCMC

Recovery of **multi-modal posterior** (signal with $\psi = -\pi/4$):



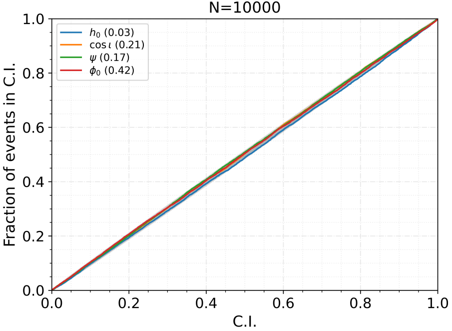
Dynesty



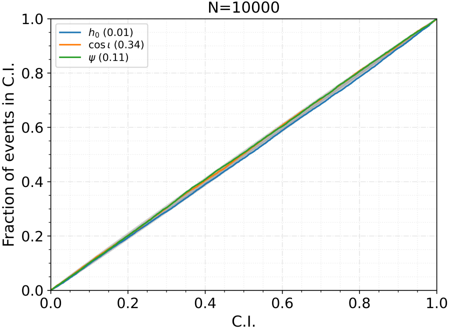
BilbyMCMC

Tests II

PP plots in Gaussian noise: ideally, $x\%$ of the total number of injections should fall in the $x\%$ credible interval



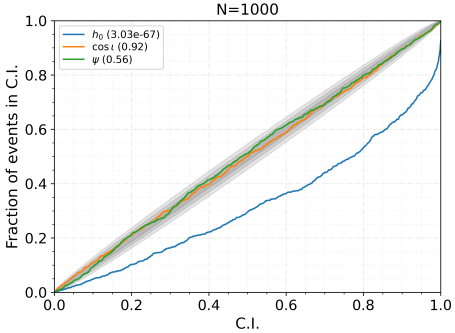
Full likelihood



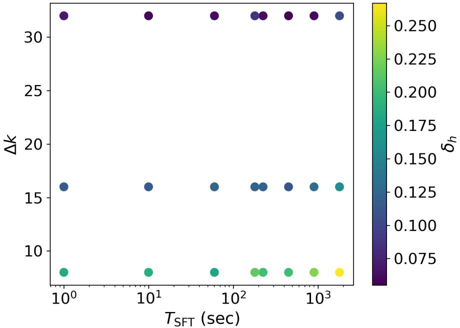
Marginalized likelihood

Tests IIb: biases

The $\mathcal{F}$-statistic calculation has some assumptions/simplifications that can lead to biases only visible for high SNR signals

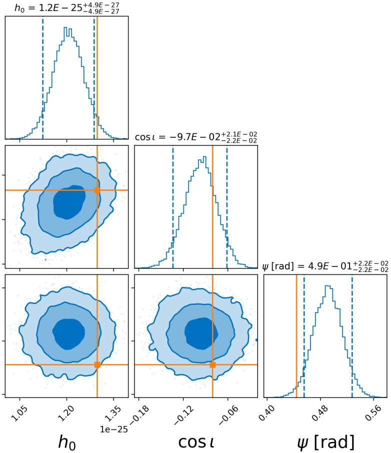


Bias in h_0 for high SNR signals

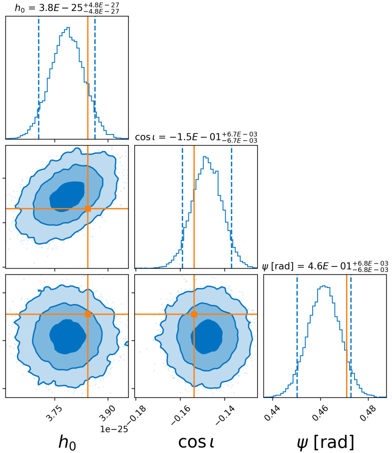


Bias as a function of some $\mathcal{F}$-statistic calculation-related parameters

Hardware injections in real data (also analyzed with the Time-domain heterodyning method for cross-check):



Hardware injection 3



Hardware injection 6

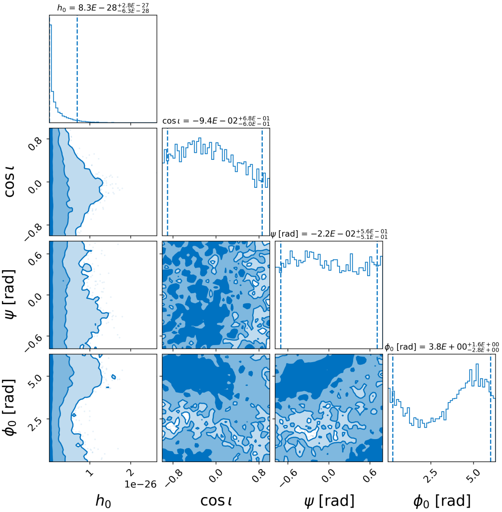
Outline

Search for PSR J1526-2744

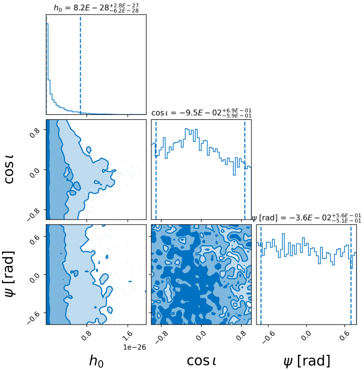
- Discovered in a **joint survey by TRAPUM and FERMI-LAT**, distance of 1.3 kpc and $f_0 = 803.5$ Hz
- Timing solution is derived from 13 years of FERMI-LAT data that overlap with the Advanced LIGO observation runs
- Search done using a **coherent combination of data from the O1, O2, and O3** observation runs of the Advanced LIGO detectors
- **No CW signal is detected**, both with the full and marginalized likelihoods
- 95% upper limit on h_0 is 6.7×10^{-27}, a factor of 9.2 larger than the spin-down upper limit of the pulsar
- 95% upper limit on ϵ is 1.3×10^{-8} assuming a moment of inertia of 10^{38} kg·m^2

Posterior distributions for PSR J1526-2744

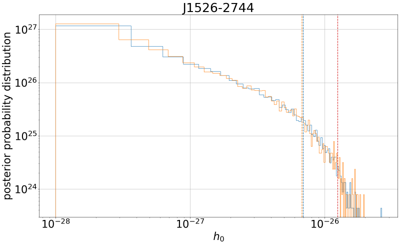
Full likelihood:



Marginalized likelihood:

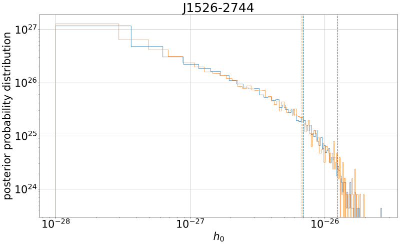


Comparison with frequentist method (in red):

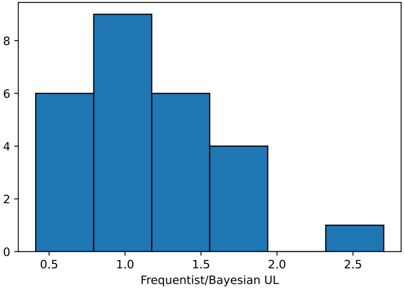


Comparison with frequentist upper limits

Comparison with frequentist method (in red):



Ratio of ULs for some other searched pulsars:



Outline

Future work

- Test limitations of the $\mathcal{F}$-statistic framework: Gaussian noise assumption, uncertainties of noise floor, Dirichlet kernel and finite SFT length, ...
- Uncertainty on phase-evolution parameters
- More complex signal models, such as signals with more than one frequency or more complicated frequency evolutions
- Characterize usage of Bayesian evidence instead of results from posterior distributions
- Code efficiency comparison to Glasgow pipeline
- Apply framework to more pulsars!

Summary

- A new Bayesian parameter estimation method, based on the highly optimized $\mathcal{F}$-statistic framework
- Tested with simulated signals, PP plots, and hardware injections
- First application on PSR J1526-2744, no detection!
- Timing solutions from new pulsars are very important

Source material

- A. Ashok et al., PRD 109, 104002 (2024)

Acknowledgements

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